Maxwell's Equations as Mean Field Equations

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Preface

The classical Maxwell's theory of electromagnetism has now been thoroughly studied for more than a century. Thus, its validity has been confirmed for a very broad spectrum of physical phenomena. However, it is known that when it comes down to the smallest constituents of the world, the classical electromagnetism is not enough: One has to turn to the quantum description.

In the beginning of the twentieth century, Planck and Einstein were the first to realize the value of the concept of a quantum. They applied this concept in explaining some, then puzzling, properties of radiation: the black body radiation and the photoelectric effect. Though, later on, the quantum ideas had been developed mostly in explaining the dynamics of atoms and electrons, that is while the dynamics of the non-relativistic particles had been explained quantum mechanically, the electromagnetic field was still being treated classically.

However, to understand certain phenomena (e.g. to obtain the right rates for spontaneous emission) it was necessary to account for the "photonic structure" of the electromagnetic field. As a result, in the late twenties, the electromagnetic field was quantized. Moreover, in the following couple of decades, the fully relativistic quantum theory of electrodynamics (QED) has emerged. The validity of this theory has been checked to an unprecedented accuracy.

So, what is the relation between the classical electromagnetism and a theory where the electromagnetic field is quantized? Intuitively it is clear that, in the regime where classical electrodynamics is valid, both relativistic and non-relativistic quantum electrodynamics have to be valid too. To be more precise, one can come up with an idea that the classical Maxwell's equations arise as a mean field limit, namely from the more fundamental equations where the Maxwell field is quantized. To explore this idea more rigorously is *the goal of this thesis*.

As a precaution, we would like to emphasize that this thesis does not contain a full rigorous derivation of the mean field limit to obtain the Maxwell's equations. The main drawback is that we treat some operators (e.g. photon field operators, the derivative operator) as being formally bounded, even though they are definitely not bounded on the full space under consideration. A possible way out of this difficulty could be to define an appropriate subspace where these operators are bounded; or maybe redefine some of these operators so as to get the required properties and show that this would create only a small error which vanishes in the mean field limit.

1 Introduction

1.1 The mean field idea

The connection between microscopic (fundamental) and macroscopic (effective) physical theories is usually not the most straightforward one. Normally, a microscopic theory is successfully applicable for systems with a small number of particles. However, with an increasing number of particles, the microscopic equations get more and more difficult to solve. Then, for some situations with a large number of particles involved, a detailed microscopic description may just unnecessarily complicate the understanding. To understand those physical situations, it is necessary to realize what the most important tendencies are. The macroscopic description just aims for that.

How do we get a macroscopic description? For the phenomena which we already observe, it is usually done by "simply" summarizing the experimental facts in an ingenious way. However, if we already have a microscopic description at hand, we can wonder if it is really fundamental. If this microscopic model is going to be qualified as fundamental, its equations have not only to be valid for systems with a small number of particles, but also for large systems. In particular, it must be at least possible to come from those microscopic to the also experimentally well-checked macroscopic equations, namely it must be possible to somehow coarse grain the microscopic and thereby get to the macroscopic.

In the mean field theory, such coarse graining is achieved by replacing pair interaction potentials with a mean field: One describes the dynamics of an arbitrary particle in an average field created by all other particles. A good illustration of the mean field idea is provided by the Hartree equation

$$i\partial_t \varphi = \left(-\Delta + v * |\varphi|^2\right) \varphi, \tag{1}$$

which is useful in describing some many-boson systems (the asterisk symbol '$*$' denotes the convolution operation). The microscopic model, which, in the large particle number limit, leads to (1) is just the following N-particle Schrödinger equation:

$$i\partial_t \Psi_N = \left(-\sum_{j=1}^{N} \Delta_j + \sum_{1\leq j<k\leq N} v_N(x_j - x_k) \right) \Psi_N. \tag{2}$$

This means that, by following one or the other recipe of taking the mean field limit, one can derive (1) from (2). In particular, one comes from the pair interaction potential v_N in (2) to the mean field potential $v * |\varphi|^2$ in (1). Of course, each of such recipes requires some assumptions to be made. One of the usual assumptions is that the interaction scales, in some way, with the particle number N (in the above example, this is indicated by the subscript '$_N$' of the potential function: v_N). However, not every assumed scaling behaviour can be physically justified.

1.2 What does it mean "to derive a mean field equation"?

It is not always clear what is meant by making, for example, a statement such as "We would like to derive this equation". Therefore, in this section, we will provide a general scheme which defines what it means to derive a mean field equation. Then, in later chapters, we will apply it for the Maxwell's equations.

Let us begin with an arbitrary microscopic model which has some general properties. Its construction is such that it gives us the time evolution of the declared family of microscopic states. We denote this family by $\mathfrak{F}_{\mathrm{m}}$ (the subscript '$_{\mathrm{m}}$' stands for "microscopic"). We also require that, if we take a state from $\mathfrak{F}_{\mathrm{m}}$ and let it evolve, it will always remain within $\mathfrak{F}_{\mathrm{m}}$. Similarly, the macroscopic model, the validity of which we want to check, is constructed so as to provide us with the time evolution of the declared family of macroscopic states. Let us denote this family by $\mathfrak{F}_{\mathrm{M}}$ (the subscript '$_{\mathrm{M}}$' stands for "macroscopic"). As before, we impose the following requirement: If we take a state from $\mathfrak{F}_{\mathrm{M}}$ and let it evolve in time, it will always remain within $\mathfrak{F}_{\mathrm{M}}$.

Even though the families $\mathfrak{F}_{\mathrm{m}}$ and $\mathfrak{F}_{\mathrm{M}}$ might be not only non-overlapping, but also of quite different nature, we want to somehow compare the states from $\mathfrak{F}_{\mathrm{m}}$ with the states from $\mathfrak{F}_{\mathrm{M}}$. For this purpose, we would like to define a comparing functional

$$\alpha : \mathfrak{F}_{\mathrm{m}} \times \mathfrak{F}_{\mathrm{M}} \to \mathbb{R}_0^+$$

with the following characterization: It has to be *appropriate from the physical point of view*. Then, if $\alpha(\mathcal{S}_{\mathrm{m}}(t), \mathcal{S}_{\mathrm{M}}(t))$ is small, we say, per definition, that the microscopic state $\mathcal{S}_{\mathrm{m}}(t) \in \mathfrak{F}_{\mathrm{m}}$ is *close* to the macroscopic state $\mathcal{S}_{\mathrm{M}}(t) \in \mathfrak{F}_{\mathrm{M}}$. However, the smallness of α at just one moment t does not tell us whether the macroscopic description of *dynamics* is valid. We say that macroscopic equations are *approximately valid* during some period of time, if α remains small during that period of time.

Next, let us move to the mean field business. To derive a mean field equation means, by definition, to take a mean field limit: $N \to \infty$. So, if we are going to perform the mean field analysis, we will have to explore how the states change when we increase the number of particles. For that purpose, first we let $\mathfrak{F}_{\mathrm{m}}$, $\mathfrak{F}_{\mathrm{M}}$ (and therefore α too) depend on the particle number N. This just indicates that, in general, states of systems with different particle numbers belong to different families. Then, for the N-particle system, the comparing functional is

$$\alpha_N : (\mathfrak{F}_{\mathrm{m}})_N \times (\mathfrak{F}_{\mathrm{M}})_N \to \mathbb{R}_0^+. \tag{3}$$

Now, we turn to our macroscopic description. For each N, we choose an initial state $\mathcal{S}_{\mathrm{M}}^N(t_0) \in (\mathfrak{F}_{\mathrm{M}})_N$ and this way construct a sequence

$$\{\mathcal{S}_{\mathrm{M}}^1(t_0) \in (\mathfrak{F}_{\mathrm{M}})_1,\ \mathcal{S}_{\mathrm{M}}^2(t_0) \in (\mathfrak{F}_{\mathrm{M}})_2,\ \ldots,\ \mathcal{S}_{\mathrm{M}}^N(t_0) \in (\mathfrak{F}_{\mathrm{M}})_N,\ \ldots\}. \tag{4}$$

The time evolution of this sequence of states is given by our mean field equations:

$$\mathcal{S}^N_{\mathrm{M}}(t_0) \xrightarrow[\text{macroscopic evolution}]{t_0 \to t} \mathcal{S}^N_{\mathrm{M}}(t).$$

Similarly, we pick a sequence of initial microscopic states

$$\{\mathcal{S}^1_{\mathrm{m}}(t_0) \in (\mathfrak{F}_{\mathrm{m}})_1,\ \mathcal{S}^2_{\mathrm{m}}(t_0) \in (\mathfrak{F}_{\mathrm{m}})_2,\ ...,\ \mathcal{S}^N_{\mathrm{m}}(t_0) \in (\mathfrak{F}_{\mathrm{m}})_N,\ ...\}, \tag{5}$$

and employ our microscopic equations to evolve these states in time

$$\mathcal{S}^N_{\mathrm{m}}(t_0) \xrightarrow[\text{microscopic evolution}]{t_0 \to t} \mathcal{S}^N_{\mathrm{m}}(t).$$

To evaluate the correctness of the macroscopic description of dynamics, we have to *assume* that the sequences (4) and (5) have been chosen so as to have the property

$$\alpha_N\big(\mathcal{S}^N_{\mathrm{m}}(t_0), \mathcal{S}^N_{\mathrm{M}}(t_0)\big) \xrightarrow{N\to\infty} 0. \tag{6}$$

If, given an arbitrary $T \in [t_0, \infty)$, we manage to prove that the condition (6) implies

$$\alpha_N\big(\mathcal{S}^N_{\mathrm{m}}(t), \mathcal{S}^N_{\mathrm{M}}(t)\big) \xrightarrow{N\to\infty} 0 \text{ uniformly in } t \in [t_0, T],$$

we say that we have *derived mean field equations* of a given macroscopic model.

In real physical situations, however, we do not have infinite numbers of particles. Thus, the whole discussion above was just a formalization of the idea, that, given a large number of particles and the initial approximate validity of a mean field description (in the sense that $\alpha_N(t_0)$ is small), this mean field description will remain, for some time, approximately valid (in the sense that the value $\alpha_N(t)$ will not become too large).

The value $\alpha_N(t)$ can be thought of as an error caused by switching from the more exact to the effective description. For some situations, it is possible to bound such errors in the following way (see e.g. [1], [2], [3]):

$$\alpha_N(t) \le C_1(t)\alpha_N(t_0) + \frac{C_2(t)}{N^a}, \tag{7}$$

where C_1 and C_2 are some positive monotonically increasing functions of time, and a some positive power of N. The above inequality is quite a good result since it says that the smaller is the initial error $\alpha_N(t_0)$ and the larger is the particle number N, the smaller will be the error $\alpha_N(t)$, for some later time moment $t > t_0$. Such a result is reasonable because it is difficult to expect that large errors could evolve into the small ones and that the mean field description could still be valid for systems of few particles.

1.3 The need of deriving Maxwell's equations

One can guess that the validity of Maxwell's equations is automatically guaranteed by quantum electrodynamics via the Heisenberg equations of motion. Indeed, in QED, one has Maxwell's equations as Heisenberg equations of motion for operator-valued fields. However, these QED equations are also valid for physical situations where the Maxwell's equations for classical fields fail. For example, while the classical Maxwell's theory does not correctly describe the photoelectric effect, the quantum theory does.

Moreover, whereas in the classical Maxwell's theory a state of the electromagnetic field is given by a configuration of classical vector fields, in quantum case it is a Fock state. Consequently, the classical and quantum descriptions of the electromagnetic field are really different. Though, there is a connection between these two descriptions. In the Fock space, one can construct a specific type of states, called *coherent* or *quasi-classical* states. When the quantum Maxwell field is in a coherent state, then its expectation value is given by the corresponding classical Maxwell field and its variation is the same as in the vacuum state, which means that this variation is as small as the "vacuum fluctuations" and, moreover, does not depend on the expected field strength. This characterization implies that, for large expectation values of field amplitudes, coherent states can be very well approximated by classical states.

However, the Fock space contains much more than just coherent states. In this sense, it is a much bigger playground for the electromagnetic field, namely the Fock space is part of a *microscopic* description, whereas classical fields belong to a *macroscopic* theory.

Even though the quasi-classical states are part of the Fock space, this does not automatically imply that the *dynamics* of classical Maxwell fields has been deduced from the dynamics of quantum Maxwell fields. To come to such a "deduction", it would be natural to try the following. First, we have to choose two models (one microscopic and one macroscopic) which describe large systems of particles coupled to the radiation field. While the microscopic model will incorporate the quantized Maxwell field, the macroscopic one will contain the classical Maxwell's equations in it. The second and the main thing to do is to show that, for a large number of particles, the two descriptions of dynamics are "close to each other". For that purpose, we should then try to follow the general procedure which has been outlined in the previous subsection.

Before turning to such a procedure, we notice that similar questions regarding the relation between classical and quantum descriptions of the electromagnetic field has been explored by Šindelka [4]. His starting point is the "exact" Hamiltonian H which contains terms with the quantized electromagnetic field. Then, to get a convenient form of H, some unitary transformations are performed. Further, after the disposal of certain terms in the transformed

Hamiltonian, a simplified Hamiltonian $\bar{H}$ is obtained. From this, the most important result follows: It is shown that $\bar{H}$ generates such dynamics that the expectation value of the quantum radiation field obeys the classical Maxwell's equation coupled to the N-particle Schrödinger equation.

The analysis which we will perform in this thesis is quite different from the one which is done in [4]. The main difference is that our derivation will be in the sense which we have described in Subsection 1.2, namely we will be doing the mean field analysis, whereas the analysis in [4] is not of this type. As a result, the derivation given in [4] is, in principle, valid for arbitrarily small particle number N, while in our work, the chosen method requires to deal with error estimates such as (7), which indicate explicitly that one is not to expect a macroscopic description to be valid for situations where the particle number N is small. For example, in the situation where only one atom and few photons are involved, one should not expect the classical Maxwell's theory to be valid.

1.4 Units and notation

We will use the Heaviside-Lorentz units with $\hbar = 1$.

Three-vectors will be written in boldface. To distinguish the electromagnetic field operators from the corresponding classical fields, they will be written with hats. In fact, the only operators that we will provide with hats will be just the quantized vector potential $\hat{\boldsymbol{A}}(\boldsymbol{x})$ and the quantized transverse electric field $\hat{\boldsymbol{E}}_\perp(\boldsymbol{x})$.

Since we do not want to confuse the indices denoting different particles with indices indicating different components of a vector, we choose the following notation. The k-th component of a field $\boldsymbol{A}$ at the position of the j-th particle will be denoted by $A_k(\boldsymbol{x}_j)$. The gradient operator with respect to the j-th particle's coordinates will be denoted by $\nabla_j := \nabla_{\boldsymbol{x}_j}$. Similarly, $\Delta_j := \Delta_{\boldsymbol{x}_j}$. The derivative with respect to the k-th component of the j-th particle's coordinates we denote by $\partial_{(\boldsymbol{x}_j)_k}$.

Repeated indices will always mean summation from 1 to 3. For example,

$$\delta^\perp_{ik}(\boldsymbol{x}-\boldsymbol{x}')A_k(\boldsymbol{x}') := \sum_{k=1}^{3} \delta^\perp_{ik}(\boldsymbol{x}-\boldsymbol{x}')A_k(\boldsymbol{x}').$$

(The definition of $\delta^\perp_{ik}$ can be found in Subsection 2.3.)

A couple of words on terminology. In this work, we will use expressions "quantum electromagnetic field", "quantum Maxwell field", "quantum radiation field" and "photon field" interchangeably (despite the fact that we will quantize only the "radiation" or "transverse" part of the electromagnetic field, while staying with the classical (unquantized) "Coulomb" or "longitudinal" part).

2 Large systems of bosons coupled to the radiation field

2.1 The selection of adequate microscopic and macroscopic models

As we have discussed in the previous section, we would like to compare the dynamics of classical Maxwell fields to the dynamics of quantum Maxwell fields. It is meaningful to make such comparison if we assume that the involved field amplitudes are big enough. This assumption can be made reasonable in the situation where the Maxwell field is coupled to a large, relatively dense system of charged particles. Then, it seems reasonable to expect the validity of the classical Maxwell's equations in the $N \to \infty$ limit.

Thus, we want to couple a large number of charged particles to their radiation field. To make things as simple as possible, we will deal with charged bosons in the non-relativistic regime. Indeed, it is easier to cope with bosons than fermions because this way we will avoid the difficulties caused by the spin-$1/2$ nature of fermions. Also, as we are going to consider the non-relativistic situation, we will not have to quantize the matter field. In addition, the non-relativistic regime will allow us to consistently remove the high energy modes of the electromagnetic field, that is to introduce a cut-off: We know that the non-relativistic description must fail if the particles are allowed to interact with photons of arbitrarily high energy. Even more importantly, in the non-relativistic situation, the particle number N is conserved, which allows us to take a well-defined $N \to \infty$ limit.

Let us now pick the microscopic model. As we have discussed above, we would like to describe the dynamics of non-relativistic bosons. Well-suited for this purpose is the model of non-relativistic quantum electrodynamics given by the spinless Pauli-Fierz Hamiltonian, or in other words, by the N-particle Schrödinger equation whose Hamiltonian contains the photon field energy operator and terms describing the interaction with the photon field. Besides, we have to select the gauge we want to work in. We choose the Coulomb gauge (both for the microscopic and the macroscopic model) since it turns out to be the simplest choice for the field quantization in a situation where particles are non-relativistic. This is because, in the non-relativistic regime, it is consistent to quantize only the radiation field, while staying with the classical Coulomb field.

As our macroscopic (effective) model we choose the Hartree equation which includes terms describing the electromagnetic interaction and is coupled to the classical Maxwell's equations via the vector potential $\boldsymbol{A}$ as well as via the empirical current and charge densities. The empirical charge density will be identified with the probability density $|\varphi|^2$ multiplied by Ne (here, φ denotes the solution of the Hartree equation). Accordingly, the empirical current density will be identified with the corresponding probability current multiplied by Ne. These identifications are motivated by the fact that, in the regime where

we have many particles whose dynamics is well described by the Hartree equation, most of these particles have to be in the state φ.

Next, we are going to present the chosen macroscopic and microscopic models together with their equations in greater detail.

2.2 Coupling Maxwell's equations to the Hartree equation

Let us begin with our macroscopic model. First, we write down the following Hartree equation for one charged boson which is in the mean field created by N other identical bosons:

$$i\partial_t \varphi = H_{\mathrm{M}}\varphi = \left[\frac{1}{2m}\left(-i\nabla - \frac{e}{c}\boldsymbol{A}\right)^2 + Ne\, U_{\mathrm{Coul}}^{\mathrm{pc}} * |\varphi|^2\right]\varphi, \tag{8}$$

where $U_{\mathrm{Coul}}^{\mathrm{pc}}(\boldsymbol{x}) := e/(4\pi|\boldsymbol{x}|)$ is the Coulomb potential of a point charge (the subscript '$_{\mathrm{M}}$' in H_{M} stands for "macroscopic"). Second, we impose the Coulomb gauge condition

$$\nabla \cdot \boldsymbol{A} = 0. \tag{9}$$

As a consequence of the self-adjointness of H_{M}, we have $\partial_t \int \mathrm{d}\boldsymbol{x}\, |\varphi|^2(\boldsymbol{x}) = 0$, i.e. we can interpret

$$\rho := Ne|\varphi|^2 \tag{10}$$

as the charge density of our N-particle system. We would like to find the current density too. To this end, we differentiate the charge density with respect to time:

$$\begin{aligned}
\frac{1}{Ne}\partial_t \rho &= \varphi\partial_t\varphi^* + \varphi^*\partial_t\varphi = i\left(\varphi(H_{\mathrm{M}}\varphi)^* - \varphi^* H_{\mathrm{M}}\varphi\right) \\
&= 2\,\mathrm{Im}\,(\varphi^* H_{\mathrm{M}}\varphi) = 2\,\mathrm{Im}\left(\varphi^*\frac{ie}{mc}\boldsymbol{A}\cdot\nabla\varphi - \varphi^*\frac{1}{2m}\Delta\varphi\right) \\
&= -\frac{1}{m}\nabla\cdot\left(\mathrm{Im}\left[\varphi^*\nabla\varphi\right] - \frac{e}{c}|\varphi|^2\boldsymbol{A}\right).
\end{aligned}$$

This leads to the continuity equation

$$\begin{aligned}
\partial_t \rho + \nabla\cdot\boldsymbol{j} &= 0, \\
\boldsymbol{j} &:= N\frac{e}{m}\left(\mathrm{Im}\left[\varphi^*\nabla\varphi\right] - \frac{e}{c}|\varphi|^2\boldsymbol{A}\right),
\end{aligned} \tag{11}$$

where $\boldsymbol{j}$ is the current density.

Now, we have to couple (8) to the Maxwell's equations

$$\partial_t \boldsymbol{B} = -c\,\nabla\times\boldsymbol{E}, \tag{12}$$

$$\partial_t \boldsymbol{E} = c\,\nabla\times\boldsymbol{B} - \boldsymbol{j}, \tag{13}$$

$$\nabla\cdot\boldsymbol{E} = \rho, \tag{14}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{15}$$

To do this, first notice that differentiating (14) with respect to time and then substituting (13) into it, we get the continuity equation. This means that it is consistent to identify ρ in (14) with the expression given by (10), and $\boldsymbol{j}$ in (13) with the expression given by (11). We perform these identifications. Second, we see in (8) and (11) that it is the vector potential $\boldsymbol{A}$ which is important in accounting for the interaction between particles and the electromagnetic field. Thus, we have to rewrite (12)-(15) in terms of potentials. To do this, we repeat an exercise from classical electrodynamics.

Remember that the homogeneous Maxwell's equations (12) and (15) imply the existence of potentials $\boldsymbol{A}$ and U in terms of which we can express the fields $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{E} = -\nabla U - \frac{1}{c}\partial_t \boldsymbol{A}, \tag{16}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{17}$$

In particular, (12) and (15) are automatically satisfied provided that potentials $\boldsymbol{A}$ and U satisfy the equations (16) and (17).

Using the Coulomb gauge condition (9), we see that the inhomogeneous Maxwell's equation (14) is satisfied given that

$$\Delta U = -\rho.$$

This is Poisson's equation, whose general solution (assuming that U vanishes at infinity) is

$$U(\boldsymbol{x}, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x}' \, \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{18}$$

which is just the Coulomb potential of our particle system. Therefore, as long as the charge distribution $\rho = Ne|\varphi|^2$ is known, (9), (16) and (18) guarantee that (14) is solved.

We still have to deal with the second inhomogeneous Maxwell's equation (13). For this, we just put (16) and (17) into (13). Making use of the vector identity $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \Delta \boldsymbol{A}$ and the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$, we obtain

$$\begin{aligned} c^{-1}\left(\partial_t^2 \boldsymbol{A} - c^2 \Delta \boldsymbol{A}\right) &= \boldsymbol{j} + \partial_t(-\nabla U) \\ &= \boldsymbol{j} + \partial_t \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x}' \, \rho(\boldsymbol{x}', t) \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \right). \end{aligned} \tag{19}$$

(To obtain the second equality, we have used (18).) The last thing we could do is the substitution of (10) and (11) into (19) to get

$$c^{-1}\left(\partial_t^2 \boldsymbol{A} - c^2 \Delta \boldsymbol{A}\right) = N\left[\frac{e}{m} \operatorname{Im}\left[\varphi^* \nabla \varphi\right] - \frac{e^2}{mc}|\varphi|^2 \boldsymbol{A} + \partial_t \left(\boldsymbol{E}^{\mathrm{pc}}_{\mathrm{Coul}} * |\varphi|^2\right)\right], \tag{20}$$

where $\boldsymbol{E}^{\mathrm{pc}}_{\mathrm{Coul}}(\boldsymbol{x}) := e\boldsymbol{x}/(4\pi|\boldsymbol{x}|^3)$ is just the Coulomb field of a point charge.

Finally, equations (8), (9) and (20) define our macroscopic model - the coupled Hartree-Maxwell equations for the N-particle system. In principle, given some *mathematically "suitable"* initial data $(\varphi(\cdot,0),\ \boldsymbol{A}(\cdot,0),\ \partial_t\boldsymbol{A}(\cdot,t)|_{t=0};\ N)$, we could solve the Hartree-Maxwell equations to obtain $(\varphi(\cdot,t),\ \boldsymbol{A}(\cdot,t),\ t \geq 0;\ N)$, where '$\cdot$' emphasizes that we consider whole configurations of the fields (not just their values at some point $\boldsymbol{x}$). After having solved for φ and $\boldsymbol{A}$, we could use (18), (16) and (17) to obtain $\boldsymbol{E}$ and $\boldsymbol{B}$.

Before turning to the microscopic model, we would like to rewrite Maxwell's equations in some other form: in terms of the fields for which, in the Coulomb gauge, the quantization procedure is the simplest. To achieve this, first we will have to introduce transverse and longitudinal vector fields. Then, we will rewrite (20) as two equations for the transverse fields $\boldsymbol{A}$ and $\boldsymbol{E}_\perp$. Additionally, we will see that those equations will be first order in time (in contrast to (20)).

2.3 Transverse and longitudinal vector fields

Let us consider an arbitrary vector field $\boldsymbol{V}(\boldsymbol{x})$ and denote its Fourier transform by $\boldsymbol{\mathcal{V}}(\boldsymbol{k})$. Then we can decompose $\boldsymbol{\mathcal{V}}(\boldsymbol{k})$ into longitudinal (parallel to $\boldsymbol{k}$) and transverse (perpendicular to $\boldsymbol{k}$) parts:

$$\boldsymbol{\mathcal{V}}(\boldsymbol{k}) = \left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\boldsymbol{\mathcal{V}}(\boldsymbol{k})\right)\frac{\boldsymbol{k}}{|\boldsymbol{k}|} + \left[\boldsymbol{\mathcal{V}}(\boldsymbol{k}) - \left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\boldsymbol{\mathcal{V}}(\boldsymbol{k})\right)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right] = \boldsymbol{\mathcal{V}}_\parallel(\boldsymbol{k}) + \boldsymbol{\mathcal{V}}_\perp(\boldsymbol{k}),$$

where

$$\boldsymbol{\mathcal{V}}_\parallel(\boldsymbol{k}) := \left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\boldsymbol{\mathcal{V}}(\boldsymbol{k})\right)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \qquad \boldsymbol{\mathcal{V}}_\perp(\boldsymbol{k}) := \left[\boldsymbol{\mathcal{V}}(\boldsymbol{k}) - \left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\boldsymbol{\mathcal{V}}(\boldsymbol{k})\right)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right]. \tag{21}$$

From this we see that

$$\begin{aligned}\boldsymbol{k}\cdot\boldsymbol{\mathcal{V}}_\perp(\boldsymbol{k}) &= 0,\\ \boldsymbol{k}\times\boldsymbol{\mathcal{V}}_\parallel(\boldsymbol{k}) &= 0.\end{aligned}$$

Multiplying the last two equations by i and then Fourier transforming them back, we obtain

$$\begin{aligned}\nabla\cdot\boldsymbol{V}_\perp(\boldsymbol{x}) &= 0,\\ \nabla\times\boldsymbol{V}_\parallel(\boldsymbol{x}) &= 0,\end{aligned}$$

which characterizes transverse and longitudinal vector fields in real space.

We would also like to obtain a more explicit expression for a transverse field in real space. For this, we look at (21) and rewrite the expression for $\boldsymbol{\mathcal{V}}_\perp$ in coordinates:

$$\mathcal{V}_{\perp i}(\boldsymbol{k}) = \sum_{j=1}^{3}\left(\delta_{ij} - \frac{k_i\,k_j}{|\boldsymbol{k}|^2}\right)\mathcal{V}_j(\boldsymbol{k}).$$

Transforming the above expression back, we get

$$V_{\perp i}(\boldsymbol{x}) = \sum_{j=1}^{3} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x}'\, \delta_{ij}^{\perp}(\boldsymbol{x}-\boldsymbol{x}')V_j(\boldsymbol{x}'), \tag{22}$$

where

$$\delta_{ij}^{\perp}(\boldsymbol{x}) := \int_{\mathbb{R}^3} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(\delta_{ij} - \frac{k_i\, k_j}{|\boldsymbol{k}|^2}\right)$$

is the so-called *"transverse" delta function.*

There is an important issue regarding the above expression. In Subsection 2.1, we have mentioned that we will be using cut-offs, which means that all integrations in reciprocal space have to be cut off at some value Λ. Thus, our working definition of the transverse delta function will be the one with a cut-off:

$$\delta_{ij}^{\perp\Lambda}(\boldsymbol{x}) := \int_{|\boldsymbol{k}|\leq\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(\delta_{ij} - \frac{k_i\, k_j}{|\boldsymbol{k}|^2}\right). \tag{23}$$

Before applying the above results for Maxwell's equations, we would like to say that more details on the transverse delta function (as well as on transverse and longitudinal fields in the context of electrodynamics) can be found in the first chapter of [5].

Now, we rewrite Maxwell's equations (12)-(13) and the equation $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ in reciprocal space (the Fourier transforms of $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{A}$ and $\boldsymbol{j}$ are denoted by the curly letters $\mathcal{B}$, $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{J}$ respectively):

$$\dot{\mathcal{B}}(\boldsymbol{k},t) = -ci\boldsymbol{k}\times\mathcal{E}(\boldsymbol{k},t), \tag{24}$$

$$\dot{\mathcal{E}}(\boldsymbol{k},t) = ci\boldsymbol{k}\times\mathcal{B}(\boldsymbol{k},t) - \mathcal{J}(\boldsymbol{k},t), \tag{25}$$

$$\mathcal{B}(\boldsymbol{k},t) = i\boldsymbol{k}\times\mathcal{A}(\boldsymbol{k},t). \tag{26}$$

Substituting (26) into (24) and using $\boldsymbol{k}\times\mathcal{E}_{\parallel}(\boldsymbol{k}) = 0$, we obtain

$$-\dot{\mathcal{A}} = c\mathcal{E}_{\perp}. \tag{27}$$

Next, we substitute (26) into (25), then separate longitudinal and transverse parts to get two equations

$$\dot{\mathcal{E}}_{\parallel}(\boldsymbol{k},t) = -\mathcal{J}_{\parallel}(\boldsymbol{k},t), \tag{28}$$

$$\dot{\mathcal{E}}_{\perp}(\boldsymbol{k},t) = -c\boldsymbol{k}\times(\boldsymbol{k}\times\mathcal{A}(\boldsymbol{k},t)) - \mathcal{J}_{\perp}(\boldsymbol{k},t). \tag{29}$$

One can show that (28) is equivalent to the continuity equation. To deal with (29), we make use of the vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\mathcal{A}) = \boldsymbol{k}(\boldsymbol{k}\cdot\mathcal{A}) - |\boldsymbol{k}|^2\mathcal{A}$ and the Coulomb gauge condition, which in reciprocal space is $i\boldsymbol{k}\cdot\mathcal{A} = 0$. As a result, we obtain

$$\dot{\mathcal{E}}_{\perp}(\boldsymbol{k},t) = c|\boldsymbol{k}|^2\mathcal{A}(\boldsymbol{k},t) - \mathcal{J}_{\perp}(\boldsymbol{k},t). \tag{30}$$

Inverse Fourier transform of (27) and (30) yields

$$-\partial_t \boldsymbol{A} = c\boldsymbol{E}_\perp, \tag{31}$$
$$-\partial_t \boldsymbol{E}_\perp = c\Delta \boldsymbol{A} + \boldsymbol{j}_\perp. \tag{32}$$

Finally, one can easily check that (31)-(32) are equivalent to (19).

2.4 List of coupled equations defining the macroscopic model

As a result of the preceding analysis of fields into transverse and longitudinal parts, we can write down the coupled Hartree-Maxwell equations (with the imposed Coulomb gauge), for a system of N identical charged bosons, in the following form (for later convenience, we provide $\boldsymbol{j}$ and ρ with the superscripts 'N'):

$$i\partial_t \varphi = H_{\mathrm{M}}\varphi = \left[\frac{1}{2m}\left(-i\nabla - \frac{e}{c}\boldsymbol{A}\right)^2 + U^{\mathrm{pc}}_{\mathrm{Coul}} * \rho^N\right]\varphi, \tag{33}$$
$$\text{where } U^{\mathrm{pc}}_{\mathrm{Coul}}(\boldsymbol{x}) = \frac{e}{4\pi|\boldsymbol{x}|},\ \ \rho^N = Ne|\varphi|^2,$$
$$\nabla \cdot \boldsymbol{A} = 0,$$
$$-\partial_t \boldsymbol{A} = c\boldsymbol{E}_\perp, \tag{34}$$
$$-\partial_t \boldsymbol{E}_\perp = c\Delta \boldsymbol{A} + \boldsymbol{j}^N_\perp, \tag{35}$$
$$\boldsymbol{j}^N = N\frac{e}{m}\left(\mathrm{Im}\left[\varphi^* \nabla\varphi\right] - \frac{e}{c}|\varphi|^2\boldsymbol{A}\right). \tag{36}$$

We remind once again that (34)-(36) are equivalent to (20). Furthermore, even though it is easier to get an explicit expression for $\boldsymbol{j}^N_\perp$ by calculating $\boldsymbol{j}^N_\parallel$ and then subtracting it from $\boldsymbol{j}^N$ (actually, we have done it, implicitly, on the right hand side of (20)), we will find it useful to express $\boldsymbol{j}^N_\perp$ via the transverse delta function (see (22)-(23)):

$$j^N_{\perp i}(\boldsymbol{x}) = \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x}'\, \delta^{\perp\Lambda}_{il}(\boldsymbol{x}-\boldsymbol{x}')j^N_l(\boldsymbol{x}'), \tag{37}$$

where (according to our conventions) repeated index l means summation from 1 to 3.

Later, we will see that the above form of our macroscopic equations is convenient when we want to compare the macroscopic model to the microscopic one.

2.5 Construction of the microscopic model

The official name of the Hamiltonian which will govern our microscopic model is the "spinless Pauli-Fierz Hamiltonian". More plainly said, our microscopic model is the non-relativistic quantum electrodynamics given by the N-particle Schrödinger equation with quantized electromagnetic field.

A standard way to "derive" the spinless Pauli-Fierz Hamiltonian, which can be found, for example, in the chapter 13 of [6], would be the following. First, one postulates the "correct" classical Lagrangian and, using the gauge freedom, drops the redundant degrees of freedom. Then, one identifies canonically conjugate pairs: position and momentum variables of each particle; and canonically conjugate field variables (up to a constant, they turn out to be $\boldsymbol{A}$ and $\boldsymbol{E}_\perp$). Second, from the Lagrangian one obtains, in a standard way, the classical Hamiltonian and replaces the classical pairs of conjugate variables by the corresponding objects from the quantum theory. Third, one usually throws away the zero point energy coming from the Coulomb self-interaction and from the photon field.

Here, we will not go through all the details of this "derivation" procedure. We will just postulate the classical Hamiltonian and make the required replacements. In particular, classical fields will be promoted to quantum field operators, whose properties we are going to discuss in some detail.

Firstly, let us again fix the Coulomb gauge. It turns out that, in the Coulomb gauge, one can separate the Coulomb energy from the energy of the radiation field. This yields the following classical Hamiltonian for a system of N identical particles coupled the radiation field:

$$\begin{aligned} H_{\mathrm{cl}} = \quad & \sum_{j=1}^{N} \frac{1}{2m} \left(\boldsymbol{p}_j - \frac{e}{c} \boldsymbol{A}(\boldsymbol{x}_j) \right)^2 \\ & + V_{\mathrm{Coul}} + \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x} \left[\boldsymbol{E}_\perp(\boldsymbol{x})^2 + (\nabla \times \boldsymbol{A}(\boldsymbol{x}))^2 \right], \end{aligned} \tag{38}$$

where

$$V_{\mathrm{Coul}}(\boldsymbol{x}_1, ..., \boldsymbol{x}_N) := \sum_{1 \leq j < k \leq N} \frac{e^2}{4\pi |\boldsymbol{x}_j - \boldsymbol{x}_k|} \tag{39}$$

is the Coulomb energy (excluding the self-energy) of the particle system, and the last term in (38) is just the energy of the transverse or radiation field.

Secondly, let us identify objects which are to be quantized and which not. From (39) we see that, in the Coulomb gauge, the scalar potential is not a real dynamical degree of freedom, since it is just a function of particle coordinates (or, more generaly, of a charge density (see (18))). This means that we do not have to quantize the scalar potential, thereby "minimizing" the quantization procedure. What we have to quantize are the real dynamical variables. In (38), these are just the canonically conjugate pairs, namely the particle positions and momenta $(\boldsymbol{x}_j, \boldsymbol{p}_j)$, $1 \leq j \leq N$, and the field variables $(\boldsymbol{A}, -c^{-1}\boldsymbol{E}_\perp)$.

Thirdly, we indicate how these objects should be quantized. Since we are dealing with the non-relativistic quantum electrodynamics, massive particles

can not be created and destroyed. Thus, in contrast to the fully relativistic theory, these particles have not been provided with the field variables. As a result, the quantization of $\boldsymbol{x}_j$ and $\boldsymbol{p}_j$ is just as in quantum mechanics: We promote $\boldsymbol{x}_j$ (without changing the notation) to the multiplication operators, and $\boldsymbol{p}_j$ to the quantum mechanical momentum operators $-i\nabla_j$. Meanwhile, the quantization of the field variables $\boldsymbol{A}$ and $\boldsymbol{E}_\perp$ is achieved by promoting them to the quantum field operators $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{E}}_\perp$, whose definitions and properties we are going to discuss next.

2.6 Quantum field operators $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{E}}_\perp$

The quantum Maxwell field is defined by the field operators

$$\hat{\boldsymbol{A}}(\boldsymbol{x}) = \sum_{\lambda=1,2} \int_{|\boldsymbol{k}|\le\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{3/2}} \sqrt{\frac{c}{2|\boldsymbol{k}|}} \boldsymbol{\varepsilon}_\lambda(\boldsymbol{k}) \left(\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, a(\boldsymbol{k},\lambda) + \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, a^\dagger(\boldsymbol{k},\lambda) \right), \tag{40}$$

$$\hat{\boldsymbol{E}}_\perp(\boldsymbol{x}) = \sum_{\lambda=1,2} \int_{|\boldsymbol{k}|\le\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{3/2}} \sqrt{\frac{c|\boldsymbol{k}|}{2}} \boldsymbol{\varepsilon}_\lambda(\boldsymbol{k}) i \left(\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, a(\boldsymbol{k},\lambda) - \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, a^\dagger(\boldsymbol{k},\lambda) \right), \tag{41}$$

where we have also introduced the ultraviolet cut-off Λ. We are not going to "derive" the above expressions (one can find some motivation for these expressions, for example in [5]). We will just remind what objects the field operators $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{E}}_\perp$ are composed of.

So, for every $|\boldsymbol{k}| \le \Lambda$, there are two polarization vectors $\boldsymbol{\varepsilon}_1(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}_2(\boldsymbol{k})$ with properties

$$|\boldsymbol{\varepsilon}_1(\boldsymbol{k})| = |\boldsymbol{\varepsilon}_2(\boldsymbol{k})| = 1, \quad \boldsymbol{\varepsilon}_1(\boldsymbol{k}) \cdot \boldsymbol{k} = \boldsymbol{\varepsilon}_2(\boldsymbol{k}) \cdot \boldsymbol{k} = \boldsymbol{\varepsilon}_1(\boldsymbol{k}) \cdot \boldsymbol{\varepsilon}_2(\boldsymbol{k}) = 0. \tag{42}$$

Next, for every combination of $|\boldsymbol{k}| \le \Lambda$ and $\lambda = 1, 2$, there is an annihilation operator $a(\boldsymbol{k},\lambda)$ and a creation operator $a^\dagger(\boldsymbol{k},\lambda)$. These operators act in the Fock space and satisfy the canonical commutation relations

$$[a(\boldsymbol{k},\lambda),\, a^\dagger(\boldsymbol{k}',\lambda')] = \delta_{\lambda\lambda'}\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{43}$$

$$[a(\boldsymbol{k},\lambda),\, a(\boldsymbol{k}',\lambda')] = 0, \quad [a^\dagger(\boldsymbol{k},\lambda),\, a^\dagger(\boldsymbol{k}',\lambda')] = 0. \tag{44}$$

Let us discuss some features of the field operators (40)-(41). One very important thing is that we have retained the transversality: $\nabla\cdot\hat{\boldsymbol{A}} = 0$, $\nabla\cdot\hat{\boldsymbol{E}}_\perp = 0$. This happened because the polarization vectors $\boldsymbol{\varepsilon}_\lambda(\boldsymbol{k})$ had been chosen to be orthogonal to $\boldsymbol{k}$.

Other relevant properties are captured by the commutator $[\hat{A}_i(\boldsymbol{x}),\, \hat{E}_{\perp j}(\boldsymbol{x}')]$, where indices $i, j = 1, 2, 3$ denote components along the three constant orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. To prepare for the calculation of this commutator, we find it useful to prove one identity first. We do it by making use of

(42):

$$\begin{aligned}
\sum_{\lambda=1,2} \varepsilon_\lambda^i(\boldsymbol{k})\varepsilon_\lambda^j(\boldsymbol{k}) &= \varepsilon_1^i(\boldsymbol{k})\varepsilon_1^j(\boldsymbol{k}) + \varepsilon_2^i(\boldsymbol{k})\varepsilon_2^j(\boldsymbol{k}) \\
&= (\boldsymbol{e}_i \cdot \boldsymbol{\varepsilon}_1(\boldsymbol{k}))\,(\boldsymbol{\varepsilon}_1(\boldsymbol{k}) \cdot \boldsymbol{e}_j) + (\boldsymbol{e}_i \cdot \boldsymbol{\varepsilon}_2(\boldsymbol{k}))\,(\boldsymbol{\varepsilon}_2(\boldsymbol{k}) \cdot \boldsymbol{e}_j) \\
&\quad + \left(\boldsymbol{e}_i \cdot \frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right)\left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \cdot \boldsymbol{e}_j\right) - \left(\boldsymbol{e}_i \cdot \frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right)\left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \cdot \boldsymbol{e}_j\right) \\
&= \boldsymbol{e}_i \cdot \boldsymbol{e}_j - \left(\boldsymbol{e}_i \cdot \frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right)\left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \cdot \boldsymbol{e}_j\right) \\
&= \delta_{ij} - \frac{k_i\, k_j}{|\boldsymbol{k}|^2}. \qquad (45)
\end{aligned}$$

Now, comes the calculation of the commutator. Using (40)-(41), (43)-(44) and (45), we get

$$\begin{aligned}
&[\hat{A}_i, (\boldsymbol{x}),\, \hat{E}_{\perp j}(\boldsymbol{x}')] \\
&= \frac{-ic}{2} \sum_{\lambda,\lambda'} \int_{|\boldsymbol{k}|\le\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \int_{|\boldsymbol{k}'|\le\Lambda} \mathrm{d}\boldsymbol{k}' \left(\frac{|\boldsymbol{k}'|}{|\boldsymbol{k}|}\right)^{1/2} \varepsilon_\lambda^i(\boldsymbol{k})\varepsilon_{\lambda'}^j(\boldsymbol{k}')\delta_{\lambda\lambda'}\delta(\boldsymbol{k}-\boldsymbol{k}')\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\boldsymbol{k}'\cdot\boldsymbol{x}'} \\
&\quad + \frac{-ic}{2} \sum_{\lambda,\lambda'} \int_{|\boldsymbol{k}|\le\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \int_{|\boldsymbol{k}'|\le\Lambda} \mathrm{d}\boldsymbol{k}' \left(\frac{|\boldsymbol{k}'|}{|\boldsymbol{k}|}\right)^{1/2} \varepsilon_\lambda^i(\boldsymbol{k})\varepsilon_{\lambda'}^j(\boldsymbol{k}')\delta_{\lambda\lambda'}\delta(\boldsymbol{k}-\boldsymbol{k}')\, \mathrm{e}^{-(i\boldsymbol{k}\cdot\boldsymbol{x} - i\boldsymbol{k}'\cdot\boldsymbol{x}')} \\
&= \frac{-ic}{2} \sum_{\lambda=1,2} \int_{|\boldsymbol{k}|\le\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \left[\varepsilon_\lambda^i(\boldsymbol{k})\varepsilon_\lambda^j(\boldsymbol{k}) + \varepsilon_\lambda^i(-\boldsymbol{k})\varepsilon_\lambda^j(-\boldsymbol{k})\right] \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\
&= -ic \int_{|\boldsymbol{k}|\le\Lambda} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \left(\delta_{ij} - \frac{k_i\, k_j}{|\boldsymbol{k}|^2}\right) \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\
&= -ic\, \delta_{ij}^{\perp\Lambda}(\boldsymbol{x}-\boldsymbol{x}'), \qquad (46)
\end{aligned}$$

where, in the second summand of the first equality, we made substitutions of the integration variables: $\boldsymbol{k} \to -\boldsymbol{k}$ and $\boldsymbol{k}' \to -\boldsymbol{k}'$. This way, we obtained the second equality. Then, to get to next line, we used (45). In the last step, we recognized the formula for the transverse delta function with a cut-off (see (23)).

2.7 Microscopic model: the spinless Pauli-Fierz Hamiltonian

We are ready to promote the classical Hamiltonian (38) to the quantum one. By replacing the classical dynamical variables with the corresponding quantum objects (see the last paragraph of Subsection 2.5) we obtain (for later convenience, we also provide H_{m}, V_{Coul} with superscripts '^{N}' and Ψ with the subscript '$_N$')

$$H_{\mathrm{m}}^N = \sum_{j=1}^{N} \frac{1}{2m} \left(-i\nabla_j - \frac{e}{c} \hat{\boldsymbol{A}}(\boldsymbol{x}_j) \right)^2 + V_{\mathrm{Coul}}^N + H_{\mathrm{f}}, \tag{47}$$

where

$$V_{\mathrm{Coul}}^N(\boldsymbol{x}_1, ..., \boldsymbol{x}_N) = \sum_{1 \le j < k \le N} \frac{e^2}{4\pi |\boldsymbol{x}_j - \boldsymbol{x}_k|} \tag{48}$$

is the Coulomb (electrostatic) interaction energy and

$$H_{\mathrm{f}} := \sum_{\lambda=1,2} \int_{|\boldsymbol{k}| \le \Lambda} \mathrm{d}\boldsymbol{k}\, c|\boldsymbol{k}| a^\dagger(\boldsymbol{k}, \lambda) a(\boldsymbol{k}, \lambda) \tag{49}$$

is the energy operator for the quantum radiation field. (We have arrived at (49) by substituting (40) and (41) into the last summand of (38). After this substitution, we dealt with the resultant expression by using (42)-(44); then, we dropped the constant vacuum energy.) Again, the subscript '$_{\mathrm{m}}$' in H_{m}^N stands for "microscopic".

Thus, our microscopic model for the system of N identical bosons coupled to the quantized radiation field is given by

$$i\partial_t \Psi_N = H_{\mathrm{m}}^N \Psi_N, \tag{50}$$

where H_{m}^N is the *spinless Pauli-Fierz Hamiltonian* (47).

3 Method of deriving mean field equations

3.1 Comparing functional α & Grönwall's lemma

In Subsection 1.2, we have outlined a general scheme for deriving mean field equations. However, to turn this into practice, a lot has to be done. In the previous section, we have already coped with one important concern, namely we have carefully chosen two models which we want to compare: one macroscopic and one microscopic. The further, most important, aspects to deal with are:

- a proper choice of the comparing functional α;
- a way of proving an inequality of type (7).

These two aspects are captured by the new method of Peter Pickl [1]. The main idea behind this method is to construct a comparing functional α_N which is *appropriate from the physical point of view* and show that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_N(t)\right| \leq c_1(t)\alpha_N(t) + \frac{c_2(t)}{N^a}, \tag{51}$$

where c_1 and c_2 are some non-negative, integrable functions of time, and a a positive power of N. The last step is to obtain, from (51), an inequality of type (7). For this, one just has to apply Grönwall's lemma, which we only state, but not prove:

Lemma 3.1 (Grönwall's lemma)**.** *Let $\beta : [0,T] \to [0,\infty)$ be a continuous function which is differentiable on $(0,T)$. Let a_1, a_2 be non-negative, integrable functions on $[0,T]$. If*

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) \leq a_1(t)\beta(t) + a_2(t),$$

then

$$\beta(t) \leq \beta(0)\,\mathrm{e}^{\int_0^t \mathrm{d}t' a_1(t')} + \int_0^t \mathrm{d}t' a_2(t')\,\mathrm{e}^{\int_{t'}^t \mathrm{d}s\, a_1(s)}$$

for all $0 \leq t \leq T$.

If, in the above lemma, a_1 and a_2 are just constants, then the resulting inequality is $\beta(t) \leq \mathrm{e}^{a_1 t}\,\beta(0) + (\mathrm{e}^{a_1 t} - 1)\,\frac{a_2}{a_1}$. We will prove this weaker version of Grönwall's lemma.

Lemma 3.2. *Let $\beta : [0,\infty) \to \mathbb{R}$ be a continuous function which is differentiable on $(0,\infty)$. If, given constants $a_1, a_2 \in (0,\infty)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t'}\beta(t') \leq a_1\beta(t') + a_2 \tag{52}$$

for all $0 \leq t' < \infty$, then

$$\beta(t) \leq \mathrm{e}^{a_1 t}\,\beta(0) + (\mathrm{e}^{a_1 t} - 1)\,\frac{a_2}{a_1}$$

for all $0 \le t < \infty$.

Proof. Multiply (52) by $\mathrm{e}^{-a_1 t'}$ to obtain

$$\mathrm{e}^{-a_1 t'} \frac{\mathrm{d}}{\mathrm{d}t'} \beta(t') \le \left(a_1 \beta(t') + a_2\right) \mathrm{e}^{-a_1 t'},$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t'}\Big(\mathrm{e}^{-a_1 t'}\, \beta(t')\Big) + a_1\, \mathrm{e}^{-a_1 t'}\, \beta(t') \le a_1 \beta(t')\, \mathrm{e}^{-a_1 t'} + a_2\, \mathrm{e}^{-a_1 t'}\,.$$

Integrate both sides of the last inequality from 0 to t to get

$$\mathrm{e}^{-a_1 t}\, \beta(t) \le \beta(0) + \frac{a_2}{a_1}(1 - \mathrm{e}^{-a_1 t}).$$

To obtain the required result, we just multiply both sides of the last inequality by $\mathrm{e}^{a_1 t}$. □

3.2 An example with the Hartree equation

We would like to give one example of application of the method which has been outlined above, namely the derivation of the Hartree equation. Since all the details of this application can be found in [1], here we will only discuss the main points of the derivation.

To begin with, the goal is to derive the Hartree equation (with spherically symmetric potential v)

$$i\partial_t \varphi = H_{\mathrm{Har}} \varphi = \left(-\Delta + v * |\varphi|^2\right) \varphi \tag{53}$$

from the N-particle Schrödinger equation

$$i\partial_t \Psi_N = H^N_{\mathrm{Schr}} \Psi_N = \left(-\sum_{j=1}^{N} \Delta_j + \sum_{1 \le j < k \le N} v_N(x_j - x_k) \right) \Psi_N, \tag{54}$$

where the interaction potential scales in the following way: $v_N = v/N$.

The Hartree equation (53) describes the dynamics of one particle which is in the mean field produced by all the other particles. It is approximately valid in those situations when almost all particles behave "in the same way", that is are "in the same state". Meanwhile, the N-particle Schrödinger equation (54) is capable of describing much more complicated situations, in particular where a significant amount of particles are "strangely entangled". The solution of (54) is an N-particle wave function Ψ^t_N. We say that particles are approximately "in the same state" if

$$\Psi^t_N(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N) \approx \prod_{j=1}^{N} \varphi^t(\boldsymbol{x}_j), \tag{55}$$

where φ^t is some one-particle wave function. We expect that φ^t should be the solution of (53).

Since there are many ways to compare functions, we have to define what we mean by '$\approx$' in (55). The tool for such comparison will be nothing else but the functional α_N, which we still have to construct. For this construction, first we need the following definition:

Definition 3.1 (Projectors p_j^φ, q_j^φ)**.** *For any combination of $\varphi \in L^2(\mathbb{R}^3)$ and $1 \le j \le N$ we define projectors*

$$\begin{aligned} p_j^\varphi : L^2(\mathbb{R}^{3N}) &\longrightarrow L^2(\mathbb{R}^{3N}) \\ \Psi_N(\boldsymbol{x}_1, ..., \boldsymbol{x}_N) &\longmapsto \varphi(\boldsymbol{x}_j) \int\limits_{\mathbb{R}^3} \varphi^*(\boldsymbol{x}_j) \Psi_N(\boldsymbol{x}_1, ...\boldsymbol{x}_j, ..., \boldsymbol{x}_N) \mathrm{d}\boldsymbol{x}_j \end{aligned}$$

and

$$\begin{aligned} q_j^\varphi : L^2(\mathbb{R}^{3N}) &\longrightarrow L^2(\mathbb{R}^{3N}) \\ \Psi_N &\longmapsto (1 - p_j^\varphi)\Psi_N. \end{aligned}$$

Further, from projectors p_j^φ and q_j^φ, we construct, for any $0 \le k \le N$, the projector (which also acts in $L^2(\mathbb{R}^{3N})$)

$$P_{N,k}^\varphi := \left(\prod_{j=1}^{k} q_j^\varphi \prod_{j=k+1}^{N} p_j^\varphi \right)_{\mathrm{sym}}, \tag{56}$$

where the subscript "sym" means symmetrization, namely $P_{N,k}^\varphi$ is given by

$$P_{N,k}^\varphi := \sum_{a \in \mathcal{A}_k} \prod_{j=1}^{N} (p_j^\varphi)^{1-a_j} (q_j^\varphi)^{a_j},$$

where

$$\mathcal{A}_k := \{(a_1, a_2, ..., a_N) |\ a_j \in \{0, 1\}, \sum_{j=1}^{N} a_j = k\}.$$

Next, we denote the scalar product on $L^2(\mathbb{R}^{3N})$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and come to the definition of the comparing functional:

Definition 3.2. *The comparing functional α_N is defined by*

$$\begin{aligned} \alpha_N :\ L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) &\longrightarrow \mathbb{R}_0^+ \\ (\Psi_N, \varphi) &\longmapsto \langle\!\langle \Psi_N, \sum_{k=0}^{N} \frac{k}{N} P_{N,k}^\varphi \Psi_N \rangle\!\rangle. \end{aligned}$$

From this definition, we see that α_N is an expectation value of "the portion of particles which are not in the state φ". This is because one can think of

$\langle\!\langle \Psi_N, P^{\varphi}_{N,k}\Psi_N\rangle\!\rangle$ as probability (see (59)-(60)) that "exactly k particles are not in the state φ". Thus, α_N given by Defintion 3.2 is our way to quantify the "distance" between Ψ_N and φ. If $\alpha_N(\Psi_N,\varphi)$ is small, we say, per definition, that the "macroscopic" state φ is close to the "microscopic" state Ψ_N:

$$\alpha_N(\Psi_N,\varphi)\approx 0 \quad \Longleftrightarrow \quad \Psi_N(\boldsymbol{x}_1,...\boldsymbol{x}_N)\approx \prod_{j=1}^{N}\varphi(\boldsymbol{x}_j). \tag{57}$$

However, as it was discussed in Subsection 1.2, we have to check if the closeness of the initial states implies the closeness of the time-evolved states. For this, we have to take the derivative of α_N with respect to time and, as it was discussed in Subsection 3.1, prove an inequality of type (51). Before doing that, let us prove the following lemma:

Lemma 3.3. *Choose $\varphi \in L^2(\mathbb{R}^3)$ and consider the comparing functional α_N given in Definition 3.2. If $\Psi_N \in L^2(\mathbb{R}^{3N})$ is symmetric, then*

$$\alpha_N(\Psi_N,\varphi) = \langle\!\langle \Psi_N, q_1^{\varphi}\Psi_N\rangle\!\rangle. \tag{58}$$

Proof. First, notice that $P^{\varphi}_{N,k}$, $0\le k\le N$, is indeed a family of projectors satisfying

$$P^{\varphi}_{N,k}P^{\varphi}_{N,j} = P^{\varphi}_{N,k}\delta_{kj}, \tag{59}$$

$$1 = \sum_{k=0}^{N} P^{\varphi}_{N,k}. \tag{60}$$

Multiply (60) by $N^{-1}\sum_{j=1}^{N} q_j^{\varphi}$ to obtain

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N} q_j^{\varphi} &= \frac{1}{N}\sum_{k=0}^{N}\left(\sum_{j=1}^{N} q_j^{\varphi}P^{\varphi}_{N,k}\right) = \frac{1}{N}\sum_{k=0}^{N}\left(\sum_{j=1}^{N} q_j^{\varphi}\left[\prod_{j'=1}^{k} q_{j'}^{\varphi}\prod_{j'=k+1}^{N} p_{j'}^{\varphi}\right]_{\mathrm{sym}}\right)\\ &= \frac{1}{N}\sum_{k=0}^{N}\left(kP^{\varphi}_{N,k}\right).\end{aligned}$$

From this, it follows that

$$\alpha_N(\Psi_N,\varphi) = \langle\!\langle \Psi_N, \frac{1}{N}\sum_{j=1}^{N} q_j^{\varphi}\Psi_N\rangle\!\rangle = \langle\!\langle \Psi_N, q_1^{\varphi}\Psi_N\rangle\!\rangle,$$

where, in the last step, we have used the symmetry of Ψ_N. □

Equation (58) provides us with a useful expression for the comparing functional α_N. Thus, we are ready to begin comparing the two descriptions of dynamics: the one given by (53) and the other by (54).

Let us *denote the solution of (53) by φ^t, and that of (54) by Ψ_N^t*, which we assume to be symmetric. Then, we take the time derivative of $\alpha_N(\Psi_N^t, \varphi^t)$. Using (54), (53) and the result of Lemma 3.3, we get (since we do not want to carry the superscripts 't', we will drop them now and restore at the end of the calculation)

$$\begin{aligned} \mathrm{d}_t\, \alpha_N(\Psi_N, \varphi) &= \mathrm{d}_t \langle\!\langle \Psi_N, q_1^\varphi \Psi_N \rangle\!\rangle \\ &= i \langle\!\langle \Psi_N, [H_{\mathrm{Schr}}^N - H_{\mathrm{Har}}^1,\, q_1^\varphi] \Psi_N \rangle\!\rangle, \end{aligned}$$

where $[\cdot,\cdot]$ is the commutator and H_{Har}^1 the Hartree Hamiltonian for the first particle. Then, noticing that q_1^φ commutes with everything which does not depend on $\boldsymbol{x}_1$ and using the symmetry of Ψ_N, we continue:

$$\begin{aligned} &= i \langle\!\langle \Psi_N, [\sum_{1\le j<k\le N} v_N(\boldsymbol{x}_j - \boldsymbol{x}_k) - v * |\varphi|^2(\boldsymbol{x}_1),\, q_1^\varphi] \Psi_N \rangle\!\rangle \qquad (61) \\ &= i \langle\!\langle \Psi_N, [\sum_{k=2}^{N} v_N(\boldsymbol{x}_1 - \boldsymbol{x}_k) - v * |\varphi|^2(\boldsymbol{x}_1),\, q_1^\varphi] \Psi_N \rangle\!\rangle \\ &= i \langle\!\langle \Psi_N, [(N-1) v_N(\boldsymbol{x}_1 - \boldsymbol{x}_2) - v * |\varphi|^2(\boldsymbol{x}_1),\, q_1^\varphi] \Psi_N \rangle\!\rangle. \end{aligned}$$

Now, we define $V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) := (N-1) v_N(\boldsymbol{x}_1 - \boldsymbol{x}_2) - v * |\varphi|^2(\boldsymbol{x}_1)$ and continue using $p_1^\varphi + q_1^\varphi = 1$, $p_2^\varphi + q_2^\varphi = 1$:

$$\begin{aligned} &= -2 \operatorname{Im} \langle\!\langle \Psi_N, V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) q_1^\varphi \Psi_N \rangle\!\rangle \\ &= -2 \operatorname{Im} \langle\!\langle \Psi_N, p_1^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) q_1^\varphi \Psi_N \rangle\!\rangle \\ &= -2 \operatorname{Im} \langle\!\langle \Psi_N, p_1^\varphi (p_2^\varphi + q_2^\varphi) V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) (p_2^\varphi + q_2^\varphi) q_1^\varphi \Psi_N \rangle\!\rangle. \qquad (62) \end{aligned}$$

Next, we should estimate (62) to get a bound of type (51). Since in this subsection it is not our goal to go into technicalities, we will just explain, heuristically, why it is possible to obtain such a bound. Then we will just cite the result from [1].

So, from (62), we get

$$\begin{aligned} |\,\mathrm{d}_t\, \alpha_N(\Psi_N, \varphi)| \le \quad & 2 |\langle\!\langle \Psi_N, p_1^\varphi p_2^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) p_2^\varphi q_1^\varphi \Psi_N \rangle\!\rangle| \\ & + 2 |\langle\!\langle \Psi_N, p_1^\varphi p_2^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) q_2^\varphi q_1^\varphi \Psi_N \rangle\!\rangle| \\ & + 2 |\operatorname{Im} \langle\!\langle \Psi_N, p_1^\varphi q_2^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) p_2^\varphi q_1^\varphi \Psi_N \rangle\!\rangle| \\ & + 2 |\langle\!\langle \Psi_N, p_1^\varphi q_2^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) q_2^\varphi q_1^\varphi \Psi_N \rangle\!\rangle|, \qquad (63) \end{aligned}$$

where we have retained "Im" in the third summand since it is easy to see that $\langle\!\langle \Psi_N, p_1^\varphi q_2^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) p_2^\varphi q_1^\varphi \Psi_N \rangle\!\rangle$ is a real quantity, which implies that the third summand is zero.

The fourth summand in (63) we bound using Schwarz inequality:

$$\begin{aligned} 2 |\langle\!\langle \Psi_N, p_1^\varphi q_2^\varphi V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) q_2^\varphi q_1^\varphi \Psi_N \rangle\!\rangle| &\le \|V_N(\boldsymbol{x}_1, \boldsymbol{x}_2) q_2^\varphi p_1^\varphi \Psi_N\| \|q_2^\varphi q_1^\varphi \Psi_N\| \\ &\le 2 (\|p_1^\varphi V_N^2(\boldsymbol{x}_1, \boldsymbol{x}_2) p_1^\varphi\|_{\mathrm{op}})^{1/2} \|q_2^\varphi \Psi_N\|^2 \\ &= 2 (\|p_1^\varphi V_N^2(\boldsymbol{x}_1, \boldsymbol{x}_2) p_1^\varphi\|_{\mathrm{op}})^{1/2} \alpha_N(\Psi_N, \varphi), \end{aligned}$$

where, to obtain the last line, we have used the symmetry of Ψ_N. It can be shown that the operator norm $\|p_1^\varphi V_N^2(\boldsymbol{x}_1,\boldsymbol{x}_2)p_1^\varphi\|_{\mathrm{op}}$ is finite for some classes of v's and φ's. Then, we can indeed bound the fourth summand by a multiple of α_N.

To deal with the second summand in (63) is more complicated since both q_1^φ and q_2^φ are on the right hand side of $V_N(\boldsymbol{x}_1,\boldsymbol{x}_2)$, which implies that we can not immediately apply Schwarz inequality, as we did in evaluating the fourth summand. Therefore, we have to "transport one of the q^φ's" to the left hand side of $V_N(\boldsymbol{x}_1,\boldsymbol{x}_2)$. It turns out that such "transportation" causes an error which is smaller than a multiple of N^{-1}, which is good enough if we aim for an inequality of type (51).

To evaluate the first summand in (63), first notice that

$$p_2^\varphi v_N(\boldsymbol{x}_1-\boldsymbol{x}_2)p_2^\varphi = p_2^\varphi(v_N * |\varphi|^2)(\boldsymbol{x}_1).$$

Then, using the scaling behaviour defined by $v_N = v/N$, it is not difficult to see that

$$2|\langle\!\langle \Psi_N, p_1^\varphi p_2^\varphi V_N(\boldsymbol{x}_1,\boldsymbol{x}_2)p_2^\varphi q_1^\varphi\Psi_N\rangle\!\rangle| = 2N^{-1}|\langle\!\langle \Psi_N, p_1^\varphi p_2^\varphi (v*|\varphi|^2)(\boldsymbol{x}_1)q_1^\varphi\Psi_N\rangle\!\rangle|.$$

It can be shown that $|\langle\!\langle \Psi_N, p_1^\varphi p_2^\varphi (v*|\varphi|^2)(\boldsymbol{x}_1)q_1^\varphi\Psi_N\rangle\!\rangle|$ can be bounded by a function of time. Thus, the whole first summand in (63) can be bounded by a multiple of N^{-1}.

Having explained, in some detail, why the time derivative of α_N can be bounded by a multiple of α_N plus a multiple of N^{-1}, let us, now, state the exact result:

Theorem 3.1. *Let $\frac{1}{r}+\frac{1}{s}=1$, $1\le r\le s\le\infty$. If $v\in L^{2r}(\mathbb{R}^3)$, $\varphi\in L^{2s}(\mathbb{R}^3)$, then*

$$|\,\mathrm{d}_t\,\alpha_N(\Psi_N,\varphi)| \le 10\|v\|_{2r}\|\varphi\|_{2s}\left(\alpha_N(\Psi_N,\varphi)+N^{-1}\right)$$

for all symmetric $\Psi_N\in L^2(\mathbb{R}^{3N})$.

Proof. See Lemma 3.2 in [1]. □

Let us bring the superscripts 't' back to the solutions Ψ_N and φ. Then, Theorem 3.1 and Lemma 3.1 imply

$$\alpha_N(\Psi_N^t,\varphi^t) \le \left(\mathrm{e}^{\int_0^t \mathrm{d}t' C(t')}\right)\alpha_N(\Psi_N^0,\varphi^0) + \left(\int_0^t \mathrm{d}t' C(t')\,\mathrm{e}^{\int_{t'}^t \mathrm{d}s\, C(s)}\right)\frac{1}{N},$$

where $C(t) := 10\|v\|_{2r}\|\varphi^t\|_{2s}$. This is what we wanted to prove.

The whole thing means that, as long as we are happy with the definition of "what it means to derive a mean field equation" (see Section 1.2) and consider our chosen comparing functional (Definition (3.2)) as physically appropriate, we have the right to say that we have derived (53) from (54) under the scaling condition $v_N = v/N$ and the conditions listed in Theorem 3.1.

4 Application of the method for Maxwell's equations

4.1 Scaling of the interaction and other assumptions

In this section, we will apply the method which we have discussed in Section 3 to compare the two models which we chose in Section 2, namely the goal is to show that, *under certain assumptions*, the model given by the Pauli-Fierz Hamiltonian (see Subsection 2.7) leads, in the mean field limit, to the Hartree-Maxwell equations (see Subsection 2.4).

Now, we would like to discuss the assumptions we make. One of the most important is the assumption regarding the scaling of the interaction. For our purposes, we choose the scaling behaviour defined in the following way:

$$e(N) = \frac{1}{\sqrt{N}}, \tag{64}$$

which means that the electron charge gets smaller with an increasing number of particles, or more properly said, the more particles we have, the weaker is the interaction between them. Hence, from the physical point of view, it is clear that as long as we take Eq. (64) alone, the only sense that can be made out of it is the following: The electron charge is fixed and (64) is just an equation for N. If we adopt this point of view, whatever we prove with the use of (64), it will be valid only for a particular number of particles N which solves (64). In addition, it seems that this could be a good starting point, since, depending on which system of units we choose, the numerical value of the electron charge is different. We know that, to define a system of units, some other physical constants, for example $\hbar$, c, have to be given numerical values too. Since c relates units of space and time, we get a hint that, in order to make sense out of (64), we should try to rescale space and time "accordingly". However, here, we are not going to try to do this.

Even though, from the physical point of view, the choice (64) is not that clear, from the mathematical point of view, nobody forbids us to make such an assumption, which turns out to be very convenient. In Subsection 1.2, we have explained that, in order to compare macroscopic and microscopic equations, we have to assume that the initial macroscopic state is close to the initial microscopic state. Moreover, since the goal is to analyse the situation when the number of particles is large, we have to construct those initial states for any particle number so that the condition (6) is valid. If, for example, part of our macroscopic description is the Hartree equation, then, with an increasing number of particles, we would like to choose the initial state φ^0 "broader and broader". This is because we do not want particles to "get packed too compactly", so that the average force which acts on a particle would not get too big; only in such regime we can expect our mean field equations to be valid. However, mathematically, a much easier way is to assume that the initial state φ^0 remains the same for any particle number, and only the initial

microscopic states are assumed to be chosen so as to have (6). Then, however, particles get packed really compactly into φ^0, and the average force acting on a particle greatly increases. A trick out of that is to scale the interaction with the number of particles, that is to make the interaction weaker with an increasing number of particles.

Notice that, even though we choose the scaling behaviour defined by (64), the current density $\boldsymbol{j}^N$ (see (36)) grows as $\sqrt{N}$. This, in turn, means that the fields $\boldsymbol{A}$ and $\boldsymbol{E}_\perp$ (see Maxwell's equations (34)-(35)) grow as $\sqrt{N}$, which we do not like since, if particles are in very strong fields, our non-relativistic description must fail. If, however, we multiply equations (34)-(36) by $e(N)$ and take $e(N)\boldsymbol{A}$, $e(N)\boldsymbol{E}_\perp$ as our new field variables, we see that the N-dependence disappears completely out of Equations (33)-(36) and, as a consequence, the fields $e(N)\boldsymbol{A}$ and $e(N)\boldsymbol{E}_\perp$ are of order 1.

As a result of the motivation given above, we will impose the scaling behaviour (64) both to our macroscopic and microscopic equations, namely we will treat e in (33)-(36) and (47)-(48) as N-dependent. Then, our new field variables will be $e(N)\boldsymbol{A}$, $e(N)\boldsymbol{E}_\perp$, $e(N)\hat{\boldsymbol{A}}$ and $e(N)\hat{\boldsymbol{E}}_\perp$. Moreover, we will formally require, without any proof, the finiteness of these operator norms:

$$\begin{aligned} &\|e(N)\boldsymbol{A}\|_{\mathrm{op}},\ \|e(N)\boldsymbol{E}_\perp\|_{\mathrm{op}}, \\ &\|e(N)\hat{\boldsymbol{A}}\|_{\mathrm{op}},\ \|e(N)\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}},\ \|\nabla\|_{\mathrm{op}} < \infty. \end{aligned} \tag{65}$$

Even though the requirement that $\|e(N)\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}}$, $\|e(N)\hat{\boldsymbol{A}}\|_{\mathrm{op}} < \infty$ is physically clear, as it has to be so on the space of states where the electromagnetic energy per particle is finite, we know that operators $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{E}}_\perp$ are not bounded on the full Fock space.

The last formal assumption is that all the classical and quantum fields are zero outside some arbitrarily large but finite volume V. This is necessary in order to make the contribution of the "vacuum fluctuations" to α_N^b and α_N^c finite (see Definition 4.1).

Because of the formal assumptions which we have discussed in the last couple of paragraphs, we point out once more, that our derivation will *not* be full and rigorous.

4.2 Definition of the comparing functional α

In the above subsection, we have imposed the scaling of the interaction (see (64)), which, to make the notation "lighter", we will keep implicit (later, at some point of our calculation, the N-dependence of e will be restored). The next thing to do is to apply the method (see Section 3) for our pair of macroscopic and microscopic models.

We begin by defining the comparing functional α_N, which will be our tool in comparing the two descriptions of dynamics:

Definition 4.1. *The comparing functional α_N is given by*

$$\alpha_N := \alpha_N^a + \alpha_N^b + \alpha_N^c,$$

where the functionals α_N^a, α_N^b and α_N^c we define as follows:

$$\begin{aligned}
\alpha_N^a :\ L^2(\mathbb{R}^{3N};\mathbb{C}) \otimes \mathcal{F} \times L^2(\mathbb{R}^3;\mathbb{C}) &\longrightarrow \mathbb{R}_0^+ \\
(\Psi_N,\, \varphi) &\longmapsto \langle\!\langle \Psi_N, \sum_{k=0}^{N} \frac{k}{N} P_{N,k}^{\varphi} \Psi_N \rangle\!\rangle, \\
\alpha_N^b :\ L^2(\mathbb{R}^{3N};\mathbb{C}) \otimes \mathcal{F} \times L^2(\mathbb{R}^3;\mathbb{R}^3) &\longrightarrow \mathbb{R}^+ \\
(\Psi_N,\, e\boldsymbol{A}) &\longmapsto \langle\!\langle \Psi_N, \left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y})\right]^2 \Psi_N \rangle\!\rangle_{;\boldsymbol{y}}, \\
\alpha_N^c :\ L^2(\mathbb{R}^{3N};\mathbb{C}) \otimes \mathcal{F} \times L^2(\mathbb{R}^3;\mathbb{R}^3) &\longrightarrow \mathbb{R}^+ \\
(\Psi_N,\, e\boldsymbol{E}_\perp) &\longmapsto \langle\!\langle \Psi_N, \left[e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y})\right]^2 \Psi_N \rangle\!\rangle_{;\boldsymbol{y}},
\end{aligned}$$

where $P_{N,k}^{\varphi}$ is given by (56), whereas $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{E}}_\perp$ by (40) resp. (41). $\mathcal{F}$ denotes the Fock space. Notation "$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{;\boldsymbol{y}}$" means integration of $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ ($\langle\!\langle \cdot,\cdot \rangle\!\rangle$ is the scalar product on $L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$) over $\boldsymbol{y}$: $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{;\boldsymbol{y}} := \int \mathrm{d}\boldsymbol{y}\, \langle\!\langle \cdot,\cdot \rangle\!\rangle$.

(In the above definition, the fact that α_N^a, α_N^b and α_N^c have different physical dimensions is, for our analysis, not that important: We can assume that the summands α_N^b and α_N^c contain constant multiplicands (of magnitude 1) which have physical dimensions such that α_N^b and α_N^c are dimensionless.)

If the $L^2(\mathbb{R}^{3N})$-part of Ψ_N is symmetric, Definition 4.1 implies the following expression for α_N:

$$\begin{aligned}
\alpha_N(\Psi_N;\, (\varphi, e\boldsymbol{A}, e\boldsymbol{E}_\perp)) = \quad & \langle\!\langle \Psi_N, q_1^{\varphi} \Psi_N \rangle\!\rangle \\
+ \quad & \langle\!\langle \Psi_N, \left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y})\right]^2 \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
+ \quad & \langle\!\langle \Psi_N, \left[e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y})\right]^2 \Psi_N \rangle\!\rangle_{;\boldsymbol{y}}, \qquad (66)
\end{aligned}$$

where we made use of Lemma 3.3 to get the expression for α_N^a. Since we are going to deal with the case where the $L^2(\mathbb{R}^{3N})$-part of Ψ_N is symmetric, *Eq. (66) will be our working definition of α_N.*

Let us explain "what α_N^a, α_N^b and α_N^c do":

- α_N^a: Notice that our definition of α_N^a is effectively the same as the one given by Defintion 3.2 (because the Fock space part of Ψ_N has no effect on α_N^a). Thus, from our discussion in the last section (see the first couple of pages of Subsection 3.2), it is clear that the smallness of $\alpha_N^a(\Psi_N, \varphi)$ means that the portion of particles which "are not in the state φ" is small.

- α_N^b, α_N^c: From the definition of α_N^b and α_N^c, we see that the smallness of $\alpha_N^b(\Psi_N, \boldsymbol{A})$ and $\alpha_N^b(\Psi_N, \boldsymbol{E}_\perp)$ implies the smallness of fluctuations of the quantum fields $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{E}}_\perp$ around the classical fields $\boldsymbol{A}$ resp. $\boldsymbol{E}_\perp$.

So far, Ψ_N and $(\varphi, e\boldsymbol{A}, e\boldsymbol{E}_\perp)$ were not necessarily the solutions of our microscopic resp. macroscopic equations. However, if we are to employ α_N for performing the "micro-macro comparison", we have to explore the time dependence of $\alpha_N(\Psi_N^t; [\varphi^t, e\boldsymbol{A}(t), e\boldsymbol{E}_\perp(t)])$, where Ψ_N^t and $[\varphi^t, e\boldsymbol{A}(t), e\boldsymbol{E}_\perp(t)]$ are solutions of (50) resp. (33)-(36). This is what we are going to do next.

4.3 Evaluation of $\mathrm{d}_t\,\alpha$

In the last subsection, we chose our comparing functional α_N and put the solutions of our equations into it. Now, as the method requires (see Section 3), we have to take the time derivative of α_N and bound it by a multiple of α_N plus a multiple of N^{-1}.

Before starting with the calculation, we would like to say that throughout the calculation we will be using the self-adjointness of $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{E}}_\perp$ and their derivatives as well as of the field energy operator H_f. Also, as we have pointed out in Subsection 4.1, all the fields vanish outside the volume V. As a result, all the boundary terms from partial integration will be always zero, which means that we will be allowed to bring the derivative operator from one side of a scalar product to the other – we will just have to accompany such an operation with a change of an overall sign.

Let us turn to the calculation. Assuming that the $L^2(\mathbb{R}^{3N})$-part of Ψ_N is symmetric and therefore using (66), we write down

$$\begin{aligned}
&\mathrm{d}_t\,\alpha_N(\Psi_N^t; [\varphi^t, e\boldsymbol{A}(t), e\boldsymbol{E}_\perp(t)]) \\
&\qquad = \mathrm{d}_t\,\alpha_N^a(\Psi_N^t;\, \varphi^t) + \mathrm{d}_t\,\alpha_N^b(\Psi_N^t;\, e\boldsymbol{A}(t)) + \mathrm{d}_t\,\alpha_N^c(\Psi_N^t;\, e\boldsymbol{E}_\perp(t)) \\
&\qquad = \mathrm{d}_t \langle\!\langle \Psi_N^t, q_1^{\varphi^t} \Psi_N^t \rangle\!\rangle + \mathrm{d}_t \langle\!\langle \Psi_N^t, \left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y}, t) \right]^2 \Psi_N^t \rangle\!\rangle_{;\boldsymbol{y}} \\
&\qquad\quad + \mathrm{d}_t \langle\!\langle \Psi_N^t, \left[e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y}, t) \right]^2 \Psi_N^t \rangle\!\rangle_{;\boldsymbol{y}}.
\end{aligned} \tag{67}$$

From now on, to ease the notation, we will omit the superscripts 't', whereas the t-dependence of $\boldsymbol{A}$ and $\boldsymbol{E}_\perp$ will be kept, for some time, explicit.

We begin with $\mathrm{d}_t\,\alpha_N^a(\Psi_N;\, \varphi)$. Using (50) and (33), we obtain

$$\begin{aligned}
\mathrm{d}_t\,\alpha_N^a(\Psi_N;\, \varphi) &= \mathrm{d}_t \langle\!\langle \Psi_N, q_1^\varphi \Psi_N \rangle\!\rangle \\
&= i \langle\!\langle \Psi_N, [H_\mathrm{m}^N,\, q_1^\varphi] \Psi_N \rangle\!\rangle + \langle\!\langle \Psi_N, (\partial_t\, q_1^\varphi)\, \Psi_N \rangle\!\rangle \\
&= i \langle\!\langle \Psi_N, [H_\mathrm{m}^N - H_\mathrm{M}^1,\, q_1^\varphi] \Psi_N \rangle\!\rangle,
\end{aligned} \tag{68}$$

where H_M^1 denotes the Hartree Hamiltonian for the first particle and H_m^N the Pauli-Fierz Hamiltonian for N particles. Let us get a more explicit version of

the commutator in (68):

$$
\begin{aligned}
&[H_{\mathrm{m}}^N - H_{\mathrm{M}}^1,\, q_1^\varphi] \\
&= \left[\frac{1}{2m}\left(-i\nabla_1 - \frac{e}{c}\hat{\boldsymbol{A}}(\boldsymbol{x}_1)\right)^2 - \frac{1}{2m}\left(-i\nabla_1 - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_1,t)\right)^2 + V_{\mathrm{Coul}}^N - U_{\mathrm{Coul}}^{\mathrm{pc}} * \rho^N,\; q_1^\varphi\right] \\
&= \frac{i}{mc}\left[e\hat{\boldsymbol{A}}(\boldsymbol{x}_1)\cdot\nabla_1 - e\boldsymbol{A}(\boldsymbol{x}_1,t)\cdot\nabla_1,\; q_1^\varphi\right] + \frac{1}{2mc^2}\left[e^2\hat{\boldsymbol{A}}^2(\boldsymbol{x}_1) - e^2\boldsymbol{A}^2(\boldsymbol{x}_1,t),\; q_1^\varphi\right] \\
&\quad + \left[V_{\mathrm{Coul}}^N - U_{\mathrm{Coul}}^{\mathrm{pc}} * \rho^N,\; q_1^\varphi\right],
\end{aligned}
$$

where, in the second step, we made use of the fact that we are working in the Coulomb gauge. Now, we rewrite (68):

$$
\begin{aligned}
\mathrm{d}_t\, \alpha_N^a(\Psi_N;\varphi) = &-\frac{2}{mc}\,\mathrm{Re}\,\langle\!\langle \Psi_N,\, \left(e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t)\right)\cdot\nabla_1\, q_1^\varphi\Psi_N\rangle\!\rangle \\
&-\frac{1}{mc^2}\,\mathrm{Im}\,\langle\!\langle \Psi_N,\, \left(e^2\hat{\boldsymbol{A}}^2(\boldsymbol{x}_1) - e^2\boldsymbol{A}^2(\boldsymbol{x}_1,t)\right) q_1^\varphi\Psi_N\rangle\!\rangle \\
&+ i\langle\!\langle \Psi_N,\, \left[V_{\mathrm{Coul}}^N - U_{\mathrm{Coul}}^{\mathrm{pc}} * \rho^N,\; q_1^\varphi\right]\Psi_N\rangle\!\rangle, \qquad (69)
\end{aligned}
$$

where, in obtaining the first summand, we have integrated by parts.

Let us first deal with the last term in (69). Using the scaling (64) for the expressions (48) and (33), we obtain:

$$
\begin{aligned}
V_{\mathrm{Coul}}^N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) &= \frac{1}{N}\sum_{1\le j<k\le N}\frac{1}{4\pi|\boldsymbol{x}_j - \boldsymbol{x}_k|}, \\
U_{\mathrm{Coul}}^{\mathrm{pc}} * \rho^N &= \frac{1}{4\pi|\cdot|} * |\varphi|^2.
\end{aligned}
$$

Then, comparing the third summand in (69) with (61) from Subsection 3.2, where the scaling condition was $v_N = v/N$, we see that they are the same if we set $v(\boldsymbol{x}) = (4\pi|\boldsymbol{x}|)^{-1}$. This means that we can apply Theorem 3.1 for the third summand in (69), since $v \in L^2 + L^\infty$:

$$
v(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|} = v_{\le 1}(\boldsymbol{x}) + v_{>1}(\boldsymbol{x}),
$$

where $v_{\le 1} \in L^2$ and $v_{>1} \in L^\infty$ are given by

$$
v_{\le 1}(\boldsymbol{x}) = \begin{cases} (4\pi|\boldsymbol{x}|)^{-1} & \text{for } |\boldsymbol{x}| \le 1, \\ 0 & \text{for } |\boldsymbol{x}| > 1; \end{cases} \qquad v_{>1}(\boldsymbol{x}) = \begin{cases} 0 & \text{for } |\boldsymbol{x}| \le 1, \\ (4\pi|\boldsymbol{x}|)^{-1} & \text{for } |\boldsymbol{x}| > 1. \end{cases}
$$

Now, we use Theorem 3.1 twice: once for $v_{\le 1}$ (with $r = 1$, $s = \infty$) and once for $v_{>1}$ (with $r = \infty$, $s = 1$). As a result, we bound the third summand in (69):

$$
\begin{aligned}
&\left|-2\,\mathrm{Im}\,\langle\!\langle \Psi_N,\, \left(V_{\mathrm{Coul}}^N - U_{\mathrm{Coul}}^{\mathrm{pc}} * \rho^N\right) q_1^\varphi\Psi_N\rangle\!\rangle\right| \\
&\qquad \le\; 10\|v_{\le 1}\|_2\|\varphi\|_\infty\left(\alpha_N^a(\Psi_N,\varphi) + N^{-1}\right) \\
&\qquad\quad + 10\|v_{>1}\|_\infty\|\varphi\|_2\left(\alpha_N^a(\Psi_N,\varphi) + N^{-1}\right).
\end{aligned}
$$

It is not difficult to check that $\|v_{\leq 1}\|_2 = (4\pi)^{-1/2}$ and $\|v_{>1}\|_\infty = (4\pi)^{-1}$. Using the normalization condition $\|\varphi\|_2 = 1$, we continue the above inequality:

$$\leq \; 3(1+\|\varphi\|_\infty)\left(\alpha_N^a(\Psi_N,\varphi) + N^{-1}\right). \tag{70}$$

Next, we turn to the first two summands in (69). Using $p_1^\varphi + q_1^\varphi = 1$, we continue the above inequality:

$$\begin{aligned}
&-\frac{2}{mc}\,\mathrm{Re}\,\langle\!\langle \Psi_N, (p_1^\varphi + q_1^\varphi)\big(e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t)\big)\cdot\nabla_1\, q_1^\varphi \Psi_N\rangle\!\rangle \\
&-\frac{1}{mc^2}\,\mathrm{Im}\,\langle\!\langle \Psi_N, (p_1^\varphi + q_1^\varphi)\big(e^2\hat{\boldsymbol{A}}^2(\boldsymbol{x}_1) - e^2\boldsymbol{A}^2(\boldsymbol{x}_1,t)\big)\, q_1^\varphi \Psi_N\rangle\!\rangle \\
&\qquad = \; -\frac{2}{mc}\,\mathrm{Re}\,\langle\!\langle \Psi_N,\, p_1^\varphi\left(e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t)\right)\cdot\nabla_1\, q_1^\varphi \Psi_N\rangle\!\rangle \\
&\qquad\quad -\frac{1}{mc^2}\,\mathrm{Im}\,\langle\!\langle \Psi_N,\, p_1^\varphi\big(e^2\hat{\boldsymbol{A}}^2(\boldsymbol{x}_1) - e^2\boldsymbol{A}^2(\boldsymbol{x}_1,t)\big)\, q_1^\varphi \Psi_N\rangle\!\rangle.
\end{aligned} \tag{71}$$

where the terms with two q_1^φ's have dropped since one of them was imaginary and the other real, whereas we had to take real resp. imaginary parts of these terms. To bound the above expression, first we evaluate the following norm (we will use the the bra-ket notation: $p_1^\varphi = |\varphi(\boldsymbol{x}_1)\rangle\langle\varphi(\boldsymbol{x}_1)|$):

$$\begin{aligned}
\|(e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t))p_1^\varphi\Psi_N\|^2 &= \langle\!\langle \Psi_N,\, p_1^\varphi\big[e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t)\big]^2 p_1^\varphi \Psi_N\rangle\!\rangle \\
&= \langle\!\langle \Psi_N|\varphi(\boldsymbol{x}_1)\rangle\langle\varphi(\boldsymbol{x}_1)|\left[e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t)\right]^2|\varphi(\boldsymbol{x}_1)\rangle\langle\varphi(\boldsymbol{x}_1)|\Psi_N\rangle\!\rangle \\
&= \langle\!\langle \Psi_N|\varphi(\boldsymbol{x}_1)\rangle\langle\varphi(\boldsymbol{y})|\left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t\right]^2|\varphi(\boldsymbol{y})\rangle\langle\varphi(\boldsymbol{x}_1)|\Psi_N\rangle\!\rangle \\
&\leq \|\varphi\|_\infty^2\langle\!\langle \Psi_N|\varphi(\boldsymbol{x}_1)\rangle\langle\varphi(\boldsymbol{x}_1)|\int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{y}\left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right]^2\Psi_N\rangle\!\rangle \\
&\leq \|\varphi\|_\infty^2\langle\!\langle \Psi_N\left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right]\cdot p_1^\varphi\left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right]\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
&\leq \|\varphi\|_\infty^2\langle\!\langle \Psi_N\left[e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right]^2\Psi_N\rangle\!\rangle_{;\boldsymbol{y}},
\end{aligned}$$

where, in the last step, we have used Schwarz inequality and the fact that $\|p_1^\varphi\|_{\mathrm{op}} = 1$. The above calculation implies that

$$\|(e\hat{\boldsymbol{A}}(\boldsymbol{x}_1) - e\boldsymbol{A}(\boldsymbol{x}_1,t))p_1^\varphi\Psi_N\| \;\leq\; \|\varphi\|_\infty\sqrt{\alpha_N^b(\Psi_N;\, e\boldsymbol{A})}. \tag{72}$$

Then, using Schwarz inequality, (72) and the assumption (65), we get the following bound of (71):

$$|(71)| \leq \frac{2}{mc}\|\varphi\|_\infty\|\nabla\|_{\mathrm{op}}\sqrt{\alpha_N^b}\sqrt{\alpha_N^a} + \frac{1}{mc^2}\|\varphi\|_\infty\|e\hat{\boldsymbol{A}} + e\boldsymbol{A}(t)\|_{\mathrm{op}}\sqrt{\alpha_N^b}\sqrt{\alpha_N^a}. \tag{73}$$

Finally, the above bound together with (70) imply

$$
\begin{aligned}
|\mathrm{d}_t\,\alpha_N^a(\Psi_N;\,\varphi)| \;\le\; & \frac{2}{mc}\|\varphi\|_\infty\|\nabla\|_{\mathrm{op}}\sqrt{\alpha_N^b}\sqrt{\alpha_N^a} \\
& + \frac{1}{mc^2}\|\varphi\|_\infty\left(\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \|e\boldsymbol{A}(t)\|_{\mathrm{op}}\right)\sqrt{\alpha_N^b}\sqrt{\alpha_N^a} \\
& + 3(1+\|\varphi\|_\infty)\left(\alpha_N^a + N^{-1}\right) \\
\le\; & \|\varphi\|_\infty\left(\frac{\|\nabla\|_{\mathrm{op}}}{mc} + \frac{\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \|e\boldsymbol{A}(t)\|_{\mathrm{op}}}{2mc^2} + 3\|\varphi\|_\infty^{-1} + 3\right)\alpha_N \qquad (74) \\
& + 3(1+\|\varphi\|_\infty)N^{-1}, \qquad (75)
\end{aligned}
$$

where we have used $2\sqrt{\alpha_N^b}\sqrt{\alpha_N^a} \le (\alpha_N^b + \alpha_N^a) \le \alpha_N$.

To evaluate $\mathrm{d}_t\,\alpha_N^b$ and $\mathrm{d}_t\,\alpha_N^c$, we will need the following commutators:

$$
\begin{aligned}
[\hat{A}_i,(\boldsymbol{x}),\,\hat{A}_j(\boldsymbol{x}')] \;&=\; [\hat{E}_{\perp i},(\boldsymbol{x}),\,\hat{E}_{\perp j}(\boldsymbol{x}')] = 0, \qquad (76)\\
[H_{\mathrm{f}},\,\hat{\boldsymbol{A}}(\boldsymbol{x})] \;&=\; ic\hat{\boldsymbol{E}}_\perp(\boldsymbol{x}), \qquad (77)\\
[H_{\mathrm{f}},\,\hat{\boldsymbol{E}}_\perp(\boldsymbol{x})] \;&=\; ic\Delta\hat{\boldsymbol{A}}(\boldsymbol{x}), \qquad (78)
\end{aligned}
$$

where H_{f} is the energy operator for the quantum radiation field (see (49)). Commutators (76)-(78) can be checked in a similar manner as the commutator $[\hat{A}_i(\boldsymbol{x}),\,\hat{E}_{\perp j}(\boldsymbol{x}')] = -ic\,\delta_{ij}^{\perp\Lambda}(\boldsymbol{x}-\boldsymbol{x}')$ (see Subsection 2.6).

Now, with the help of Heisenberg equation of motion for operators, we write down $\mathrm{d}_t\,\alpha_N^b(\Psi_N;\,e\boldsymbol{A})$:

$$
\begin{aligned}
\mathrm{d}_t\,\alpha_N^b(\Psi_N;\,e\boldsymbol{A}(t)) \;&=\; \mathrm{d}_t\langle\!\langle\Psi_N,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
&=\; i\langle\!\langle\Psi_N,\left[H_{\mathrm{m}}^N,\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2\right]\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
&\quad -2e\langle\!\langle\Psi_N,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)\cdot\partial_t\boldsymbol{A}(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
&=\; i\langle\!\langle\Psi_N,\left[H_{\mathrm{m}}^N,\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2\right]\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
&\quad +2c\langle\!\langle\Psi_N,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)\cdot e\boldsymbol{E}_\perp(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}, \qquad (79)
\end{aligned}
$$

where, to get the last line, we have used the Maxwell equation (34). Further, we take care of the commutator in (79):

$$
\begin{aligned}
\left[H_{\mathrm{m}}^N,\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2\right] = & -\frac{1}{2m}[\sum_{j=1}^N\Delta_j,\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2] \\
& + \frac{ie}{mc}[\sum_{j=1}^N\hat{\boldsymbol{A}}(\boldsymbol{x}_j)\cdot\nabla_j,\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2] \\
& + \frac{e^2}{2mc^2}[\sum_{j=1}^N\hat{\boldsymbol{A}}^2(\boldsymbol{x}_j),\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2] \\
& + [H_{\mathrm{f}},\,\left(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\right)^2]. \qquad (80)
\end{aligned}
$$

It is easy to see that the first three summands in the above formula are zero. This happens because of (76) and the fact that Δ_j and ∇_j commute with everything which is not a function of $\boldsymbol{x}_j$, $1 \le j \le N$. The last line of (80) we rewrite using $[B, C^2] = [B, C]C + C[B, C]$ (where B and C are any operators) and Eq. (77):

$$\begin{aligned}
[H_{\mathrm{f}},\ e\hat{\boldsymbol{A}} - e\boldsymbol{A}]\cdot(e\hat{\boldsymbol{A}} - e\boldsymbol{A}) + (e\hat{\boldsymbol{A}} - e\boldsymbol{A}) \cdot [H_{\mathrm{f}},\ e\hat{\boldsymbol{A}} - e\boldsymbol{A}] \\
&= [H_{\mathrm{f}},\ e\hat{\boldsymbol{A}}] \cdot (e\hat{\boldsymbol{A}} - e\boldsymbol{A}) + (e\hat{\boldsymbol{A}} - e\boldsymbol{A}) \cdot [H_{\mathrm{f}},\ e\hat{\boldsymbol{A}}] \\
&= ice\hat{\boldsymbol{E}}_\perp \cdot (e\hat{\boldsymbol{A}} - e\boldsymbol{A}) + ic(e\hat{\boldsymbol{A}} - e\boldsymbol{A}) \cdot e\hat{\boldsymbol{E}}_\perp,
\end{aligned}$$

which we insert into the place of the commutator in Eq. (79). This gives us

$$\begin{aligned}
|\,\mathrm{d}_t\, \alpha_N^b(\Psi_N;\, e\boldsymbol{A}(t))| \ &= |-2c\,\mathrm{Re}\,\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)\big) \cdot e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y})\Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&\quad + 2c\,\langle\!\langle \Psi_N, (e\hat{\boldsymbol{A}}(\boldsymbol{y}) - e\boldsymbol{A}(\boldsymbol{y},t)) \cdot e\boldsymbol{E}_\perp(\boldsymbol{y},t)\Psi_N \rangle\!\rangle_{;\boldsymbol{y}}| \\
&= |2c\,\mathrm{Re}\,\langle\!\langle \Psi_N, (e\hat{\boldsymbol{A}} - e\boldsymbol{A})(\boldsymbol{y},t) \cdot (e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp)(\boldsymbol{y},t)\Psi_N \rangle\!\rangle_{;\boldsymbol{y}}| \\
&\le\ 2c\sqrt{\alpha_N^b}\sqrt{\alpha_N^c} \le c(\alpha_N^b + \alpha_N^c) \le c\alpha_N,
\end{aligned} \tag{81}$$

where we have used Schwarz inequality.

Let us turn to $\mathrm{d}_t\, \alpha_N^c(\Psi_N;\, e\boldsymbol{E}_\perp)$. Just as in (79), we make use of Heisenberg equation of motion to obtain

$$\begin{aligned}
\mathrm{d}_t\, \alpha_N^c(\Psi_N; e\boldsymbol{E}_\perp(t)) &= \mathrm{d}_t \langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2 \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&= \quad i\langle\!\langle \Psi_N, \Big[H_{\mathrm{m}}^N,\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2\Big] \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&\quad -2e\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big) \cdot \partial_t \boldsymbol{E}_\perp(\boldsymbol{y},t)\Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&= \quad i\langle\!\langle \Psi_N, \Big[H_{\mathrm{m}}^N,\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2\Big] \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&\quad +2e\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big) \cdot \big(c(\Delta\boldsymbol{A})(\boldsymbol{y},t) + \boldsymbol{j}_\perp^N(\boldsymbol{y},t)\big) \Psi_N \rangle\!\rangle_{;\boldsymbol{y}},
\end{aligned} \tag{82}$$

where, in the last step, we have used the Maxwell equation (35). Next, we have to deal with the commutator in (82):

$$\begin{aligned}
\Big[H_{\mathrm{m}}^N,\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2\Big] \\
&= -\frac{1}{2m}[\sum_{j=1}^N \Delta_j,\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2] \\
&\quad + \frac{ie}{mc}[\sum_{j=1}^N \hat{\boldsymbol{A}}(\boldsymbol{x}_j) \cdot \nabla_j,\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2] \\
&\quad + \frac{e^2}{2mc^2}[\sum_{j=1}^N \hat{\boldsymbol{A}}^2(\boldsymbol{x}_j),\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2] \\
&\quad + [H_{\mathrm{f}},\ \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)^2],
\end{aligned} \tag{83}$$

where the first summand is zero. If we insert the last term of the above equation into its place in Eq. (82), then, using (78), we obtain

$$\begin{aligned}-2\,\mathrm{Im}\,\langle\!\langle \Psi_N, &(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp)(\boldsymbol{y},t)\cdot[H_\mathrm{f},\ e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y})]\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &= -2ec\,\mathrm{Re}\,\langle\!\langle \Psi_N, (e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp)(\boldsymbol{y},t)\cdot(\Delta\hat{\boldsymbol{A}})(\boldsymbol{y})\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}. \qquad (84)\end{aligned}$$

Since $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{E}}_\perp$ do not commute with each other, the second and the third summands in Eq. (83) are not zero. Inserting these summands into their places in Eq. (82), we get the following contribution to the commutator term in Eq. (82):

$$\begin{aligned}&\frac{-2e}{mc}\sum_{j=1}^{N}\mathrm{Re}\,\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)\cdot\big[\hat{A}_k(\boldsymbol{x}_j)\partial_{(\boldsymbol{x}_j)_k},\ e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y})\big]\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &\frac{-e^2}{mc^2}\sum_{j=1}^{N}\mathrm{Im}\,\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)\cdot\Big[\hat{\boldsymbol{A}}^2(\boldsymbol{x}_j),\ e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y})\Big]\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}.\end{aligned}$$

Using $[\hat{A}_k(\boldsymbol{x}),\ \hat{E}_{\perp i}(\boldsymbol{x}')] = -ic\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}-\boldsymbol{x}')$ (see (46)), we continue:

$$\begin{aligned}&= \frac{-2e^2}{m}\sum_{j=1}^{N}\mathrm{Im}\,\langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_j-\boldsymbol{y})\partial_{(\boldsymbol{x}_j)_k}\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &\quad\frac{+2e^3}{mc}\sum_{j=1}^{N}\mathrm{Re}\,\langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_j-\boldsymbol{y})\hat{A}_k(\boldsymbol{x}_j)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &= \frac{-2e^2}{m}N\,\mathrm{Im}\,\langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y})\partial_{(\boldsymbol{x}_1)_k}\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &\quad\frac{+2e^3}{mc}N\,\mathrm{Re}\,\langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y})\hat{A}_k(\boldsymbol{x}_1)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}, \qquad (85)\end{aligned}$$

where, to get the last two lines, we have used the symmetry of the $L^2(\mathbb{R}^{3N})$-part of Ψ_N.

Now, we can rewrite the whole $\mathrm{d}_t\,\alpha^c_N$. The commutator term in (82) equals (84) + (85). Combining the 'Δ' term from the last line of Eq. (82) with (84), we obtain

$$\begin{aligned}&\mathrm{d}_t\,\alpha^c_N(\Psi_N;\, e\boldsymbol{E}_\perp(t))\\ &\quad= -2c\,\mathrm{Re}\,\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{y},t)\cdot\big[\Delta(e\hat{\boldsymbol{A}} - e\boldsymbol{A})\big](\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &\quad\frac{-2e^2}{m}N\,\mathrm{Im}\,\langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y})\partial_{(\boldsymbol{x}_1)_k}\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &\quad\frac{+2e^3}{mc}N\,\mathrm{Re}\,\langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\,\delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y})\hat{A}_k(\boldsymbol{x}_1)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}\\ &\qquad+2e\,\langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\big)\cdot\boldsymbol{j}^N_\perp(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}. \qquad (86)\end{aligned}$$

First, using Schwarz inequality, we bound the first summand of the above equation:

$$\begin{aligned}
&|-2c\,\mathrm{Re}\,\langle\!\langle \Psi_N, \left(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\right)(\boldsymbol{y},t)\cdot\left[\Delta(e\hat{\boldsymbol{A}}-e\boldsymbol{A})\right](\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}| \\
&\qquad \le 2c\sqrt{\alpha_N^c}\langle\!\langle \Psi_N, \left[\Delta(e\hat{\boldsymbol{A}}-e\boldsymbol{A})\right]^2(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}^{1/2} \\
&\qquad \le 2c\sqrt{\alpha_N^c}\Lambda^2\langle\!\langle \Psi_N, (e\hat{\boldsymbol{A}}-e\boldsymbol{A})^2(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}^{1/2} \\
&\qquad = 2c\sqrt{\alpha_N^c}\Lambda^2\sqrt{\alpha_N^b} \;\le\; c\Lambda^2(\alpha_N^c+\alpha_N^b) \;\le\; c\Lambda^2\alpha_N, \qquad (87)
\end{aligned}$$

where, in the second step we have used the fact that all our fields are with a cut-off Λ (see, for example, (40)).

Let us, now, deal with the last three lines of (86). First, from (36) and (37), we have the following expression for $\boldsymbol{j}_\perp^N$:

$$j_{\perp i}^N(\boldsymbol{y},t) = N\frac{e}{m}\int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{x_1}\,\delta_{ki}^{\perp\Lambda}(\boldsymbol{x}_1-\boldsymbol{y})\left(\mathrm{Im}\left[\varphi^*\,\partial_k\varphi\right] - \frac{e}{c}|\varphi|^2 A_k\right)(\boldsymbol{x_1},t).$$

Then, using $p_1^\varphi + q_1^\varphi = 1$ (we remind that p_1^φ and q_1^φ are given by Definition 3.1), we can write the last line of (86) as

$$\begin{aligned}
&2e\,\langle\!\langle \Psi_N, \left(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\right)\cdot\boldsymbol{j}_\perp^N(\boldsymbol{y},t)q_1^\varphi\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
+N\frac{2e^2}{m}\,&\mathrm{Im}\,\langle\!\langle \Psi_N, \left(e\hat{E}_{\perp i} - eE_{\perp i}\right)(\boldsymbol{y},t)\,p_1^\varphi\, I_{1i}(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}} \\
-N\frac{2e^3}{mc}\,&\langle\!\langle \Psi_N, \left(e\hat{E}_{\perp i} - eE_{\perp i}\right)(\boldsymbol{y},t)\,p_1^\varphi\, I_{2i}(\boldsymbol{y},t)\Psi_N\rangle\!\rangle_{;\boldsymbol{y}}, \qquad (88)
\end{aligned}$$

where

$$\begin{aligned}
I_{1i}(\boldsymbol{y},t) &:= \int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{x_1}\,\delta_{ki}^{\perp\Lambda}(\boldsymbol{x}_1-\boldsymbol{y})\left[\varphi^*\,\partial_k\varphi\right](\boldsymbol{x}_1,t), \\
I_{2i}(\boldsymbol{y},t) &:= \int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{x_1}\,\delta_{ki}^{\perp\Lambda}(\boldsymbol{x}_1-\boldsymbol{y})\left[|\varphi|^2 A_k\right](\boldsymbol{x}_1,t).
\end{aligned}$$

Next, it is not difficult to see that (recalling the bra-ket notation: $p_1^\varphi = |\varphi(\boldsymbol{x}_1)\rangle\langle\varphi(\boldsymbol{x}_1)|$)

$$\begin{aligned}
p_1^\varphi I_{1i}(\boldsymbol{y},t) &= p_1^\varphi\,\delta_{ki}^{\perp\Lambda}(\boldsymbol{x}_1-\boldsymbol{y})\partial_{(\boldsymbol{x}_1)_k}\,p_1^\varphi, \\
p_1^\varphi I_{2i}(\boldsymbol{y},t) &= p_1^\varphi\,\delta_{ki}^{\perp\Lambda}(\boldsymbol{x}_1-\boldsymbol{y})A_k(\boldsymbol{x}_1,t)\,p_1^\varphi,
\end{aligned}$$

with the help of which we rewrite the second and the third summands of (86)

together with the last two terms of (88):

$$
\begin{aligned}
&\frac{-2e^2}{m} N \operatorname{Im} \langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\, \delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y}) \partial_{(\boldsymbol{x}_1)_k} \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&\frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i}(\boldsymbol{y}) - eE_{\perp i}(\boldsymbol{y},t)\big)\, \delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y}) \hat{A}_k(\boldsymbol{x}_1) \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
+N&\frac{2e^2}{m} \operatorname{Im} \langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i} - eE_{\perp i}\big)(\boldsymbol{y},t)\, p_1^\varphi\, \delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y}) \partial_{(\boldsymbol{x}_1)_k} p_1^\varphi \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
-N&\frac{2e^3}{mc} \langle\!\langle \Psi_N, \big(e\hat{E}_{\perp i} - eE_{\perp i}\big)(\boldsymbol{y},t)\, p_1^\varphi\, \delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y}) A_k(\boldsymbol{x}_1,t)\, p_1^\varphi \Psi_N \rangle\!\rangle_{;\boldsymbol{y}} \\
&= \frac{-2e^2}{m} N \operatorname{Im} \langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \nabla_1 \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \hat{\boldsymbol{A}}(\boldsymbol{x}_1) \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^2}{m} N \operatorname{Im} \langle\!\langle \Psi_N, p_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \nabla_1 p_1^\varphi \Psi_N \rangle\!\rangle \\
&\quad -\frac{2e^3}{mc} N \langle\!\langle \Psi_N, p_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \boldsymbol{A}(\boldsymbol{x}_1,t)\, p_1^\varphi \Psi_N \rangle\!\rangle, \qquad (89)
\end{aligned}
$$

where we made use of the fact that $\hat{\boldsymbol{E}}_\perp$ and $\boldsymbol{E}_\perp$ are transverse fields, which implies (recall Eq. (22))

$$
\int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{y}\, \delta^{\perp\Lambda}_{ki}(\boldsymbol{x}_1-\boldsymbol{y}) \big(e\hat{E}_{\perp i} - eE_{\perp i}\big)(\boldsymbol{y},t) = \big(e\hat{E}_{\perp k} - eE_{\perp k}\big)(\boldsymbol{x}_1,t).
$$

Now, to continue with (89), we insert $p_1^\varphi + q_1^\varphi = 1$ next to each factor of Ψ_N in the first two summands of (89):

$$
\begin{aligned}
&= \frac{-4e^2}{m} N \operatorname{Im} \langle\!\langle \Psi_N, p_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \nabla_1 q_1^\varphi\, \Psi_N \rangle\!\rangle \\
&\quad \frac{-2e^2}{m} N \operatorname{Im} \langle\!\langle \Psi_N, q_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \nabla_1 q_1^\varphi\, \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, p_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \big(\hat{\boldsymbol{A}} - \boldsymbol{A}\big)(\boldsymbol{x}_1,t)\, p_1^\varphi\, \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, p_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \hat{\boldsymbol{A}}(\boldsymbol{x}_1)\, q_1^\varphi\, \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, q_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \big(\hat{\boldsymbol{A}} - \boldsymbol{A}\big)(\boldsymbol{x}_1,t)\, p_1^\varphi\, \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, q_1^\varphi\, \boldsymbol{A}(\boldsymbol{x}_1,t) \cdot \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t)\, p_1^\varphi\, \Psi_N \rangle\!\rangle \\
&\quad \frac{+2e^3}{mc} N \operatorname{Re} \langle\!\langle \Psi_N, q_1^\varphi \big(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp\big)(\boldsymbol{x}_1,t) \cdot \hat{\boldsymbol{A}}(\boldsymbol{x}_1)\, q_1^\varphi\, \Psi_N \rangle\!\rangle, \qquad (90)
\end{aligned}
$$

where, to obtain the first term, we have integrated by parts and made use of the fact that the transversality of $\hat{\boldsymbol{E}}_\perp$ and $\boldsymbol{E}_\perp$ implies $(e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp) \cdot \nabla f = \nabla \cdot (e\hat{\boldsymbol{E}}_\perp f - e\boldsymbol{E}_\perp f)$, for any function f.

Further, we use the scaling condition $e^2 = N^{-1}$, Schwarz inequality and Eq. (72) to bound (90):

$$\begin{aligned}|(90)| \;\leq\; & \frac{4}{m}\|\varphi\|_\infty\|\nabla\|_{\mathrm{op}}\sqrt{\alpha_N^c}\sqrt{\alpha_N^a} \;+\; \frac{2}{m}\|e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\|\nabla\|_{\mathrm{op}}\alpha_N^a \\ & + \frac{2}{mc}\|\varphi\|_\infty^2\sqrt{\alpha_N^c}\sqrt{\alpha_N^b} \;+\; \frac{2}{mc}\|\varphi\|_\infty\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}}\sqrt{\alpha_N^c}\sqrt{\alpha_N^a} \\ & + \frac{2}{mc}\|\varphi\|_\infty\|e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\sqrt{\alpha_N^a}\sqrt{\alpha_N^b} \\ & + \frac{2}{mc}\|\varphi\|_\infty\|e\boldsymbol{A}(t)\|_{\mathrm{op}}\sqrt{\alpha_N^a}\sqrt{\alpha_N^c} \;+\; \frac{2}{mc}\|e\hat{\boldsymbol{E}}_\perp - e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}}\alpha_N^a,\end{aligned} \tag{91}$$

where we have also used

$$\|(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{x}_1) - e\boldsymbol{E}_\perp(\boldsymbol{x}_1,t))p_1^\varphi\Psi_N\| \;\leq\; \|\varphi\|_\infty\sqrt{\alpha_N^c(\Psi_N;\, e\boldsymbol{E}_\perp)},$$

which is obtained in the same way as (72).

The last object we have to bound is the first line of (88). Making use of the fact that $\int \mathrm{d}\boldsymbol{y}\, \boldsymbol{V}_\perp(\boldsymbol{y})\cdot\boldsymbol{U}_\parallel(\boldsymbol{y}) = 0$ (where $\boldsymbol{V}_\perp$ is some transverse and $\boldsymbol{U}_\parallel$ some longitudinal vector field), we obtain

$$\begin{aligned}&\left|2e\left\langle\!\!\left\langle \Psi_N,\, \left(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\right)\cdot\boldsymbol{j}_\perp^N(\boldsymbol{y},t)q_1^\varphi\Psi_N\right\rangle\!\!\right\rangle_{;\boldsymbol{y}}\right| \\ &\quad= \left|2e\left\langle\!\!\left\langle \Psi_N,\, \left(e\hat{\boldsymbol{E}}_\perp(\boldsymbol{y}) - e\boldsymbol{E}_\perp(\boldsymbol{y},t)\right)\cdot\boldsymbol{j}^N(\boldsymbol{y},t)q_1^\varphi\Psi_N\right\rangle\!\!\right\rangle_{;\boldsymbol{y}}\right| \\ &\quad\leq 2\sqrt{\alpha_N^c}\sqrt{\alpha_N^a}\|e\boldsymbol{j}^N(t)\|_2 \;\leq\; (\alpha_N^c+\alpha_N^a)\|e\boldsymbol{j}^N(t)\|_2 \;\leq\; \|e\boldsymbol{j}^N(t)\|_2\alpha_N \\ &\quad\leq \frac{1}{m}\|\varphi\|_\infty\left(\|\nabla\varphi\|_2 + \frac{1}{c}\|\varphi\|_\infty\|e\boldsymbol{A}(t)\|_2\right)\alpha_N,\end{aligned} \tag{92}$$

where we have used Schwarz inequality and the following bound of $\|e\boldsymbol{j}^N(t)\|_2$ (see (36) for the definition of $\boldsymbol{j}^N$):

$$\|e\boldsymbol{j}^N(t)\|_2 \leq \frac{1}{m}\|\varphi\|_\infty\left(\|\nabla\varphi\|_2 + \frac{1}{c}\|\varphi\|_\infty\|e\boldsymbol{A}(t)\|_2\right), \tag{93}$$

which does not depend on N anymore because the scaling condition $e^2 = N^{-1}$ has been imposed.

Now, adding inequalities (87), (91) and (92), we bound $\mathrm{d}_t\,\alpha_N^c$:

$$\begin{aligned}&|\,\mathrm{d}_t\,\alpha_N^c(\Psi_N; e\boldsymbol{E}_\perp(t))| \\ &\quad\leq \Bigg(c\Lambda^2 + \frac{2}{m}\|\varphi\|_\infty\|\nabla\|_{\mathrm{op}} \;+\; \frac{2}{m}\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}}\|\nabla\|_{\mathrm{op}} + \frac{2}{m}\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\|\nabla\|_{\mathrm{op}} \\ &\qquad + \frac{1}{mc}\|\varphi\|_\infty^2 \;+\; \frac{1}{mc}\|\varphi\|_\infty\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \frac{1}{mc}\|\varphi\|_\infty\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}} \\ &\qquad + \frac{1}{mc}\|\varphi\|_\infty\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}} + \frac{1}{mc}\|\varphi\|_\infty\|e\boldsymbol{A}(t)\|_{\mathrm{op}} \;+\; \frac{2}{mc}\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}}\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} \\ &\qquad + \frac{2}{mc}\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \frac{1}{m}\|\varphi\|_\infty\|\nabla\varphi\|_2 + \frac{1}{mc}\|\varphi\|_\infty^2\|e\boldsymbol{A}(t)\|_2\Bigg)\alpha_N,\end{aligned} \tag{94}$$

where (as before) we have used $2\sqrt{\alpha_N^i}\sqrt{\alpha_N^j} \leq \alpha_N$, for $i \neq j$.

Finally, looking at inequalities (74), (81) and (94) and restoring the superscripts 't', we find that

$$\begin{aligned}
&|\,\mathrm{d}_t\,\alpha_N(\Psi_N^t;\,[\varphi^t,\,e\boldsymbol{A}(t),e\boldsymbol{E}_\perp(t)])| \\
&\qquad\leq\;\; C_1(t)\,\alpha_N(\Psi_N^t;\,[\varphi^t,\,e\boldsymbol{A}(t),\,e\boldsymbol{E}_\perp(t)]) + C_2(t)N^{-1} \\
&\qquad=\;\; C_1\Big(\Lambda,\,\|\varphi^t\|_\infty,\,\|\nabla\varphi^t\|_2,\,\|\nabla\|_{\mathrm{op}},\,\|e\boldsymbol{A}(t)\|_2, \\
&\qquad\qquad\qquad \|e\boldsymbol{A}(t)\|_{\mathrm{op}},\,\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}},\,\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}},\,\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}}\Big)\,\alpha_N(t) \\
&\qquad\quad +C_2\big(\|\varphi^t\|_\infty\big)N^{-1},
\end{aligned} \tag{95}$$

where

$$\begin{aligned}
C_1(t) \;:=\;& \frac{1}{mc}\|\varphi^t\|_\infty\|\nabla\|_{\mathrm{op}} + \frac{1}{2mc^2}\|\varphi^t\|_\infty\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \frac{1}{2mc^2}\|\varphi^t\|_\infty\|e\boldsymbol{A}(t)\|_{\mathrm{op}} \\
&+ 3\|\varphi^t\|_\infty + 3 + c + c\Lambda^2 \\
&+ \frac{2}{m}\|\varphi^t\|_\infty\|\nabla\|_{\mathrm{op}} \;+\; \frac{2}{m}\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}}\|\nabla\|_{\mathrm{op}} + \frac{2}{m}\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\|\nabla\|_{\mathrm{op}} \\
&+ \frac{1}{mc}\|\varphi^t\|_\infty^2 \;+\; \frac{1}{mc}\|\varphi^t\|_\infty\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \frac{1}{mc}\|\varphi^t\|_\infty\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}} \\
&+ \frac{1}{mc}\|\varphi^t\|_\infty\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}} + \frac{1}{mc}\|\varphi^t\|_\infty\|e\boldsymbol{A}(t)\|_{\mathrm{op}} \;+\; \frac{2}{mc}\|e\hat{\boldsymbol{E}}_\perp\|_{\mathrm{op}}\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} \\
&+ \frac{2}{mc}\|e\boldsymbol{E}_\perp(t)\|_{\mathrm{op}}\|e\hat{\boldsymbol{A}}\|_{\mathrm{op}} + \frac{1}{m}\|\varphi^t\|_\infty\|\nabla\varphi^t\|_2 + \frac{1}{mc}\|\varphi^t\|_\infty^2\|e\boldsymbol{A}(t)\|_2
\end{aligned} \tag{96}$$

and

$$C_2(t) \;:=\; 3 + 3\|\varphi^t\|_\infty. \tag{97}$$

The inequality (95) is just what we have aimed for. Lastly, we can apply Grönwall's lemma (see Lemma 3.1) and obtain

$$\begin{aligned}
\alpha_N(\Psi_N^t;\,[\varphi^t,\,e\boldsymbol{A}(t),\,e\boldsymbol{E}_\perp(t)]) \leq\;& \Big(\mathrm{e}^{\int_0^t \mathrm{d}t' C_1(t')}\Big)\,\alpha_N(\Psi_N^0;\,[\varphi^0,\,e\boldsymbol{A}(0),\,e\boldsymbol{E}_\perp(0)]) \\
&+\left(\int_0^t \mathrm{d}t' C_2(t')\,\mathrm{e}^{\int_{t'}^t \mathrm{d}s\, C_1(s)}\right)\frac{1}{N},
\end{aligned} \tag{98}$$

where C_1 and C_2 are given by (96) resp. (97).

5 Conclusions

The main result of this thesis is inequality (98), which, in the sense discussed in Subsection 1.2, can be viewed as the derivation of the classical Maxwell's equations from the more fundamental theory, namely from the non-relativistic quantum electrodynamics given by the Pauli-Fierz Hamiltonian (see Subsection 2.7). A more precise interpretation of inequality (98) is that it gives an estimate of the error (which, for some time moment t, is given by the value of the comparing functional $\alpha_N(t)$) made by switching from the quantum to the classical description of electrodynamics.

From (96), we see that this error can be made smaller by making the cut-off Λ and the kinetic energy $\|\nabla\varphi\|_2$ smaller, which is very natural since even the non-relativistic QED can be valid only provided that the photon energies as well as the kinetic energies of our massive bosons do not get too large. Similar role is also of the assumption (65) about the fields, which should be looked upon as the requirement that the average electromagnetic field energy per particle must be not too large.

From the mathematical point of view, the main drawback of our calculation are the formal assumptions (65). These assumptions have to be proven. As we have mentioned in Preface, this could possibly be achieved by properly defining a subspace where these "problematic" operators are bounded, or maybe by some "suitable redefinition" of the field operators. Another issue is the formal assumption that the field operators are zero outside some finite volume V, which is necessary so as to avoid an infinite contribution of the "vacuum fluctuations" to α_N^b and α_N^c. This could be resolved either by consistently performing the whole analysis for a finite volume, or maybe by a suitable modification of the comparing functionals α_N^b and α_N^c.

From the physical point of view, one of the most problematical issues is the scaling of the interaction (see (64)). This certainly requires a much better understanding and motivation.

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